

STOCHASTIC FEYNMAN RULES FOR YANG-MILLS THEORY ON THE PLANE

TIMOTHY NGUYEN

ABSTRACT. We analyze quantum Yang-Mills theory on \mathbb{R}^2 using a novel discretization method based on an algebraic analogue of stochastic calculus. Such an analogue involves working with “Gaussian” free fields whose covariance matrix is indefinite rather than positive definite. Specifically, we work with Lie-algebra valued fields on a lattice and exploit an approximate gauge-invariance that is restored when taking the continuum limit. This analysis is applied to show the equivalence between Wilson loop expectations computed using partial axial-gauge, complete axial-gauge, and the Migdal-Witten lattice formulation. As a consequence, we obtain intriguing Lie-theoretic identities involving heat kernels and iterated integrals.

CONTENTS

1. Introduction	1
2. The Setup	8
2.1. Generalized White-Noise	10
3. Algebraic Stochastic Calculus	12
4. Proofs of Main Theorems	27
References	30

1. INTRODUCTION

Quantum Yang-Mills theory on \mathbb{R}^2 is exactly soluble due to the fact that the theory becomes free in (complete) axial-gauge. More precisely, working in a gauge in which the connection $A = A_x dx + A_y dy$ has $A_y \equiv 0$ and A_x vanishes on the x -axis, the Yang-Mills action becomes purely quadratic. This allows for a rigorous interpretation of the Yang-Mills measure as a Gaussian measure, or more precisely, a (Lie-algebra valued) white-noise measure [8]. As a result, Wilson loop expectation values can be analyzed in terms of stochastic holonomy [11], whereby random connections are distributed according to the Yang-Mills measure. These resulting stochastic computations of [8] agree with formulas arising from the Migdal-Witten lattice formulation of Yang-Mills theory [13, 20] which uses the heat kernel action and which has been extensively studied from a variety of perspectives [1, 6, 9, 10, 12, 18, 19]. In contrast to [8] however, the standard method of evaluating expectation values of observables against Gaussian measures is to apply Wick’s Theorem, i.e., to evaluate and sum over Feynman diagrams. The Feynman diagrammatic method has

Date: October 17, 2016.

the advantage that it readily carries over to interacting quantum field theories, whereas the exact (i.e. nonperturbative) methods using white-noise analysis does not. Moreover, the conceptual requirements needed to implement Feynman diagrams, such as those pertaining to gauge-fixing and gauge-invariance, provide a rich and important arena for mathematical analysis. Among the gauge-fixing procedures most relevant to two-dimensional Yang-Mills theory are the complete axial-gauge and the partial axial-gauge, where the latter imposes that only A_y vanish. (Partial axial-gauge has the advantage that translation-invariance is manifestly preserved, whereas it must be established with considerable work in the case of complete axial-gauge [8, 11].)

The purpose of this paper is to show that Wilson loop expectations computed using Feynman diagrams in complete and partial axial-gauge agree with those computed using stochastic/lattice methods. In particular, partial axial-gauge and complete axial-gauge are equivalent. From a physical standpoint, this is to be expected since physical quantities should be independent of the choice of gauge. Mathematically however, such reasoning cannot be applied so straightforwardly owing to the fact that the use of gauge-transformations when working with low regularity random connections requires considerable care. In fact, the main difficulty runs much deeper, since the partial axial-gauge does not even give rise to an honest Gaussian measure since the connection is massless. Our initial attempt to circumvent this problem was to insert a mass regulator and then send the mass to zero, but this turns out to introduce difficulties we were not able to surmount, see Remark 4.2. So instead, we forgo measure theory and directly consider computations in the partial axial-gauge as defined purely algebraically using the Wick procedure. In this way, we are naturally led to develop an algebraic analogue of (discretized) white-noise analysis and stochastic calculus that is able to handle the partial axial-gauge. The use of gauge-invariance in this setting turns out to require significant finesse, as will become apparent later on. Our final result, which computes Wilson loop expectations in two different ways, is in purely mathematical terms a collection of nontrivial identities between integrals of heat kernels on the gauge group with iterated integrals along contours defining our Wilson loops. The ability to evaluate iterated integrals plays an important role in a variety of mathematical and quantum field theoretic settings [5]. Our own motivation stems from an investigation into the fundamental mathematical structure of quantum gauge theories, see the discussion at the end of the introduction.

We now describe our main results, with a more complete discussion of the setup given in Section 2. Fix any compact Lie group G for our gauge group. A Wilson loop observable $W_{f,\gamma}$ takes a (sufficiently smooth) connection A , computes its holonomy $\text{hol}_\gamma(A)$ about a (piecewise C^1) closed curve γ , and then applies the conjugation-invariant function $f : G \rightarrow \mathbb{C}$ to this group-valued element:

$$W_{f,\gamma}(A) := f(\text{hol}_\gamma(A)). \quad (1.1)$$

Using the path-ordered exponential representation of $\text{hol}_\gamma(A)$, we can express $W_{f,\gamma}(A)$ as a power series functional in A via (2.11).

From this representation, we can compute Wilson loop expectations, term by term, with respect to the (putative) Yang-Mills measure in partial or complete axial-gauge. This computation makes use of the Wick rule, which determines the expectation of any polynomial from the two point function (i.e. the expectation of a quadratic polynomial), see (3.2). The

use of the Wick rule means that we do not need an honest measure to compute expectations, though the expectation comes from a Gaussian measure in case the two-point function defines a positive-definite pairing. The two-point function in partial and complete axial-gauge is obtained as follows. First, we determine the corresponding (gauge-dependent) Green's functions for the Yang-Mills kinetic operator $-\partial_y^2$. Next, the integral kernels of these Green's functions give rise to corresponding propagators P_{pax} and P_{ax} in partial and complete axial-gauge, respectively, see (2.8–2.9). Finally, the insertion of these propagators into bilinear expressions of the connection A determines the two-point function, i.e., the “Feynman rules”.

From these propagators, we can attempt to make the following formal definitions

$$\begin{aligned} \langle W_{f,\gamma} \rangle_{pax}, \langle W_{f,\gamma} \rangle_{ax} \text{ “=” sum over all Feynman integrals obtained from} \\ \text{inserting propagators } P_{pax} \text{ (resp. } P_{ax}) \text{ weighted} \\ \text{by } \lambda \text{ into (2.11) using the Wick rule.} \end{aligned} \quad (1.2)$$

(For those unfamiliar with the Feynman diagram procedure implicit on the right-hand side above, see (2.12) for an explicit formula.) The weight λ is a parameter which counts the number of propagator insertions; in path integral notation, it coincides with the Yang-Mills coupling constant in (2.3).

The reason the above “definition” is formal is that the propagators (2.8) and (2.9) are singular and so do not a priori yield well-defined integral expressions in the evaluation of (1.2). This turns out not to be a problem for partial axial-gauge, since the y -dependent part of P_{pax} vanishes along the diagonal and so “cancels” a δ -function in the x -direction. But this being not the case for P_{ax} , the definition (1.2) is ill-defined because while the classical integrals occurring in (2.11) are given by Riemann integrals of a smooth connection that do not depend on how the integral is discretized, the quantum expectation values of these integrals in complete axial-gauge *depends on the discretization method*. This is most easily illustrated by noting that

$$\int_{1 > t_2 > t_1 > 0} \delta(t_2 - t_1) dt_2 dt_1 = 0 \quad (1.3)$$

whereas

$$\int_{1 \geq t_2 \geq t_1 \geq 0} \delta(t_2 - t_1) dt_2 dt_1 = \int_0^1 dt_1 = 1. \quad (1.4)$$

Hence, owing to the fact that the integrand $\delta(t_2 - t_1)$ is singular, how one approximates the domain of integration $\{1 \geq t_2 \geq t_1 \geq 0\}$, whether from below or above or an average of the two, affects the result of the integration. (Though if one weights the above integrands with a continuous function that vanishes at $t_1 = t_2$, there would be no ambiguity.)

The resolution of this predicament is that the Wilson loop expectation in complete axial-gauge is not an operation on an underlying classical observable (in this case a Riemann integral). Rather, the Wilson loop expectation is an expectation of a *stochastic integral*. There are two common constructions for stochastic integrals: the Ito integral, which uses the left-endpoint rule and the Stratonovich rule which uses the midpoint rule. They yield different answers in a manner analogous to the above computation.

Consequently, the most natural way to define $\langle W_{f,\gamma} \rangle_{ax}$ is by regarding the integrals occurring in the path ordered exponential (2.11) as iterated Stratonovich integrals of the nonsmooth random connections A . These integrals then become random matrix-valued elements, from which we can apply $f = \text{tr}$ and take the (stochastic) expectation to obtain the final numerical result. The use of the Stratonovich integral is most natural since this integral obeys the usual calculus rules under changes of variables and the like (the Ito integral does not).

We now have three ways to compute Wilson loop expectations. We have $\langle W_{f,\gamma} \rangle_{pax}$ and $\langle W_{f,\gamma} \rangle_{ax}$, both of which are computed “Feynman diagrammatically”. (For the complete axial-gauge, the expectation of iterated Stratonovich integrals can be computed by a Wick procedure which converts Stratonovich integrals to Ito-Riemann integrals, see Lemma 3.8.) They are a priori formal power series in the coupling constant λ . We also have the exact expectation $\langle W_{f,\gamma} \rangle$, which is computed using the Midgald-Witten heat kernel action on a lattice. It is the exact expectation because such a lattice formulation is invariant under subdivision and so represents an exact expectation of $W_{f,\gamma}$ in the continuum limit. The quantity $\langle W_{f,\gamma} \rangle$ is a well-defined function of λ . For us, we can take the stochastic expression (4.8) as the definition of $\langle W_{f,\gamma} \rangle$ (the work of [8] implies that this definition is equivalent to the more commonly used definition in terms of heat kernels).

Our first result amounts to a basic unraveling of stochastic constructions:

Theorem 1. *We have*

$$\langle W_{f,\gamma} \rangle = \langle W_{f,\gamma} \rangle_{ax} \quad (1.5)$$

in the sense that the power series in λ on the right-hand side is equal to the function of λ on the left-hand side.

Our main result however concerns the equivalence between partial axial-gauge and complete axial-gauge:

Theorem 2. *We have*

$$\langle W_{f,\gamma} \rangle_{pax} = \langle W_{f,\gamma} \rangle_{ax}. \quad (1.6)$$

The above results and their proofs go through unchanged if we consider products of Wilson loop observables.

Our proof of Theorem 2 requires new ideas. This is because as mentioned before, partial axial-gauge does not arise from a measure since the corresponding two-point function defines an indefinite pairing. The approach we develop is to simply push stochastic analytic constructions in the indefinite setting. This requires a psychological adjustment, akin to working with anticommuting Grassman variables instead of ordinary commuting variables. In the indefinite setting, “random variables” no longer have values, i.e. they are not functions defined on a measure space. Instead, they form an algebra that is equipped with an expectation operator defined by use of the Wick rule. Hence, we regard the analysis we develop as an *algebraic stochastic calculus*. A notable feature of this calculus is that because the associated two-point function (i.e. covariance) is no longer positive, one must forgo most basic tools for estimates such as the Cauchy-Schwarz inequality.

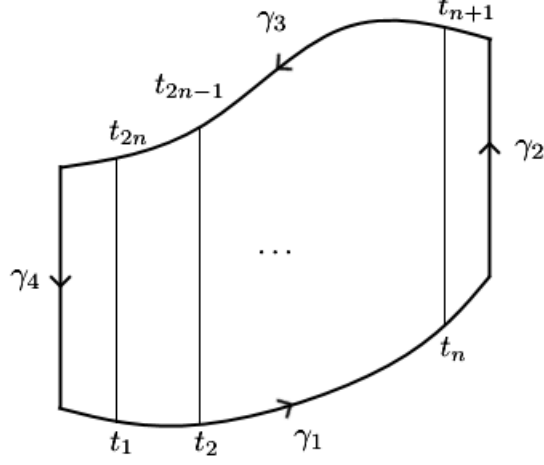


FIGURE 1. Propagators joining $(t_1, t_{2n}), (t_2, t_{2n-1}), \dots, (t_n, t_{n+1})$ in the order λ^n term of $\langle W_{f,\gamma} \rangle_{pax}$.

Example: We provide the simplest explicit example of the equality $\langle W_{f,\gamma} \rangle = \langle W_{f,\gamma} \rangle_{pax}$ in order to illustrate the type of identities our theorems provide. These identities become highly nontrivial as the complexity of γ increases.

Let $f = \text{tr}\rho(\cdot)$ be trace in some irreducible representation ρ of our gauge group G . Let γ be a curve given by joining two disjoint horizontal curves (see Definition 2.1) by vertical segments, see Figure 1. Let R denote the region it encloses and $|R|$ its area. In this case, we can easily verify (1.6) to all orders in λ as follows:

To compute $\langle W_{f,\gamma} \rangle_{pax}$, write $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ as the concatenation of paths $\gamma_4 \cdot \gamma_3 \cdot \gamma_2 \cdot \gamma_1$, where (see Figure 1)

- γ_1 is the graph of $\gamma_- : [t_0, T] \rightarrow \mathbb{R}$ traversed from t_0 to T ;
- γ_3 is the graph of $\gamma_+ : [t_0, T] \rightarrow \mathbb{R}$ traversed from T to t_0 ;
- $\gamma_-(t) < \gamma_+(t)$ for all t ;
- γ_2 and γ_4 are upward and downward moving vertical segments, joining the γ_1 and γ_3 .

In computing $\langle W_{f,\gamma} \rangle_{pax}$, the order λ^n term of $\langle W_{f,\gamma} \rangle_{pax}$ comes from inserting n copies of P_{pax} given by (2.8) into the order $2n$ term of (2.11). When we Wick contract a pair of points on γ at times $t_i < t_j$, we only pick up a nonvanishing term when we contract a point of γ_1 with that of γ_3 . (In partial axial-gauge, the propagator has only has dx -components and so integrates trivially along vertical segments; moreover, the δ -constraint in the x -direction means we cannot contract pairs of points belonging to the same γ_i). Each such Wick contraction yields both the term

$$- \left[- \frac{|\gamma_+(x(t_j)) - \gamma_-(x(t_i))|}{2} \delta(x(t_i) - x(t_j)) \right], \quad (1.7)$$

where $x(t)$ denotes the x -coordinate of $\gamma(t)$, and an insertion of $e_a e_a$ (corresponding to the identity tensor on \mathfrak{g} , where e_a is an orthonormal basis for the Lie algebra \mathfrak{g}), where we implicitly sum over the repeated index a . We pick up a factor of -1 in (1.7) since γ_3 and γ_1 go in opposite directions.

As a result, we need to choose all possible groupings of the $2n$ path ordered points t_1, \dots, t_{2n} into n pairs, each such pair determining the location of a propagator insertion. The δ -constraint in the x -direction in the propagator means that the only nonvanishing pairings arise from the choice $(t_1, t_{2n}), (t_2, t_{2n-1}), \dots, (t_n, t_{n+1})$, with t_1, \dots, t_n belonging to the domain of γ_1 and t_{n+1}, \dots, t_{2n} belonging to the domain of γ_3 . Thus the λ^n coefficient of $\langle W_{f,\gamma} \rangle_{pax}$ is given by

$$\begin{aligned} & \frac{1}{2^n} \int_{1 \geq t_{2n} \geq t_{2n-1} \geq \dots \geq t_1 \geq 0} \left[\left(\gamma_+(x(t_{2n})) - \gamma_-(x(t_1)) \right) \cdots \left(\gamma_+(x(t_{n+1})) - \gamma_-(x(t_n)) \right) \right] \times \\ & \left[\delta(x(t_{2n}) - x(t_1)) \cdots \delta(x(t_{n+1}) - x(t_n)) \right] \text{tr} \left(\rho(e_{a_1}) \cdots \rho(e_{a_n}) \rho(e_{a_n}) \cdots \rho(e_{a_1}) \right) \prod dt_i \\ & = \frac{[\text{tr}(\rho(e_a) \rho(e_a))]^n}{2^n} \int_{T \geq x_n \geq \dots \geq x_1 \geq t_0} (\gamma_+(x_1) - \gamma_-(x_1)) \cdots (\gamma_+(x_n) - \gamma_-(x_n)) \prod dx_i \\ & = \frac{c_2(\rho)^n}{2^n} \frac{1}{n!} \int_{t_0}^T (\gamma_+(x) - \gamma_-(x))^n dx \\ & = \frac{c_2(\rho)^n |R|^n}{2^n n!}, \end{aligned}$$

where $c_2(\rho) = \text{tr}(\rho(e_a) \rho(e_a))$ denotes the quadratic Casimir for the representation ρ . In going from the first line to the second, we used that $e_a e_a$ is a central element in the universal enveloping algebra of \mathfrak{g} .

On the other hand, we have

$$\langle W_{f,\gamma} \rangle = \int_G \text{tr} \rho(g) K_{\lambda|R|}(g) dg \quad (1.8)$$

$$= e^{-\lambda|R|c_2(\rho)/2} \quad (1.9)$$

where $K_t(g)$ is the convolution kernel for the heat operator $e^{-t\Delta/2}$ on G (with respect to Riemannian metric induced by the inner product on \mathfrak{g}). Indeed, the rule for computing $\langle W_{f,\gamma} \rangle$ involves placing a heat kernel at every bounded face of $\mathbb{R}^2 \setminus \gamma$ at time equal to the coupling constant λ times the area enclosed [8, 12]. The argument of each heat kernel and of f is formed from words formed out of group elements labeling the edges of γ . In the simplest case of a simple closed curve, we can treat all of γ as a single edge and so we have just a single group element $g \in G$ in (1.8), and f and $K_{\lambda|R|}$ are evaluated on g .

Thus, for every n , the order λ^n terms of $\langle W_{f,\gamma} \rangle$ and $\langle W_{f,\gamma} \rangle_{pax}$ agree. The explicit evaluation of $\langle W_{f,\gamma} \rangle_{ax}$ is a bit more involved. See Remark 4.3 for details.

Outline of paper and further remarks:

This paper is organized as follows. In Section 2, we describe a more detailed setup of the axial gauges and their propagators, as well as some terminology concerning our decomposition of Wilson loops into simpler pieces. We also discuss how maps generalizing white-noise naturally arise when performing parallel transport in axial gauges. Since this generalization involves forgoing the usual positive-definiteness conditions in probability theory, in Section 3 we develop an algebraic stochastic calculus to analyze such generalized white-noise. It involves a discretization procedure which approximates stochastic integrals by finite sums, with the latter being well-defined when forgoing measure theory. Finally, we apply these results to prove our main theorems in Section 4.

We would like to make some remarks on how our work fits into the greater context of quantum field theory. First, we note that our discretization method appears to be completely new. The standard approach to discretizing gauge theories involves placing group valued, not Lie-algebra valued fields, on a lattice. Such a discretization has a built in gauge-invariance. For us, our use of gauge-invariance only holds asymptotically, i.e., it is restored in the continuum limit. Next, we make a general observation concerning the dichotomy between two different approaches to quantum field theory. On the one hand, there has been the long-standing constructivist school which aims to construct honest measures for defining path integrals. On the other hand, because this task is plagued with many difficulties, it has been fashionable for many decades to instead understand what kind of mathematics one can derive from purely formal aspects of path integrals (with the many works of Witten being the pinnacle of such endeavors). A remarkable aspect of our work is that it provides a kind of mysterious link between the formal (not arising from a measure) aspects of partial axial-gauge to the measure-theoretic aspects of complete axial-gauge. We believe this connection deserves to be better understood. Finally, we believe the algebraic stochastic calculus we formulated is worthy of being further developed, not only for its own mathematical sake, but because it may be useful in other quantum field theoretic settings in which one has two-point functions that are not positive.

This paper is an output of the author's investigation into the relationship between the perturbative and the exact formulation of two-dimensional Yang-Mills theory [16, 17]. In these works, holomorphic gauge instead of partial axial-gauge is studied, where holomorphic gauge is regarded as a “generalized axial-gauge” distinct from the axial gauges considered here¹. In fact, it is shown that holomorphic gauge is inequivalent to the axial gauges considered here, which suggests that our main theorems concerning various equivalences are not to be taken for granted. Furthermore, [17] shows that Wilson loop expectations in holomorphic gauge also yield a set of remarkable identities relating iterated integrals and matrix integrals. We hope the stochastic analysis we provide here, which yields identities to all orders in perturbation theory, may be useful in other (quantum field-theoretic) contexts.

¹One can regard the partial axial-gauge and complete axial-gauges here as “stochastic axial-gauge”, since the two gauges are equivalent and complete axial-gauge involves a “stochastic regulator”. On the other hand, generalized axial-gauge is a generalization of Wu-Mandelstam-Liebrandt light cone gauge, which regulates partial axial-gauge using a different regulator. See [16, 17] for further details.

2. THE SETUP

Fix a compact Lie group G equipped with an ad-invariant inner product $\langle \cdot, \cdot \rangle$ on its Lie algebra \mathfrak{g} . On \mathbb{R}^2 , given a (smooth) connection $A = A_x dx + A_y dy$, which is an element of $\Omega^1(\mathbb{R}^2; \mathfrak{g})$, the space of \mathfrak{g} -valued 1-forms, we can always find a gauge transformation that places A in *partial axial-gauge*:

$$A \in \mathcal{A}_{pax} = \{A \in \Omega^1(\mathbb{R}^2; \mathfrak{g}) : A_y \equiv 0\}. \quad (2.1)$$

Having done so, we still have gauge freedom in the x -direction to place a connection in *complete axial-gauge*:

$$A \in \mathcal{A}_{ax} = \{A \in \Omega^1(\mathbb{R}^2; \mathfrak{g}) : A_y \equiv 0, A_x(\cdot, 0) \equiv 0\}. \quad (2.2)$$

This completely eliminates all gauge freedom arising from the action of the group of gauge transformations that are fixed to be the identity at the origin.

In either of these gauges, the (Euclidean) Yang-Mills path integral can be formally written as²

$$\int dA_x e^{-\frac{1}{2\lambda} \int dxdy \langle \partial_y A_x, \partial_y A_x \rangle}. \quad (2.3)$$

where A_x ranges over either \mathcal{A}_{pax} or \mathcal{A}_{ax} and λ is a coupling constant. Indeed, in these axial gauges, the curvature of the connection A is simply $\partial_y A_x dy \wedge dx$. The integrand of (2.3) is the putative Yang-Mills measure.

The Wick rule we obtain from (2.3) is determined by the Green's function we choose for the kinetic operator $-\partial_y^2$ occurring in (2.3). A Green's function satisfies

$$-\partial_y^2 G(x, y; x', y') = \delta(x - x') \delta(y - y'). \quad (2.4)$$

For partial axial-gauge, we choose the unique solution to (2.4) which is invariant under translations and reflections and homogeneous under scaling. Indeed, these are symmetries of the kinetic operator, and so we may as well impose them on the Green's function. We obtain

$$G_{pax} = \bar{G}_{pax}(y, y') \delta(x - x') \quad (2.5)$$

where

$$\bar{G}_{pax}(y, y') = -\frac{|y - y'|}{2}. \quad (2.6)$$

For complete axial-gauge, we consider $-\partial_y^2$ acting on functions which vanish along the axis $y = 0$. We impose the same symmetries as before, only now we must relinquish translation invariance in the y -direction. We thus obtain the Green's operator

$$\bar{G}_{ax}(y, y') = \begin{cases} \min(|y|, |y'|) & yy' \geq 0 \\ 0 & yy' < 0. \end{cases} \quad (2.7)$$

²The Faddeev-Popov determinant is constant in axial gauges and hence can be dropped in the path integral.

From the partial axial-gauge and complete axial-gauge Green's function, we obtain a corresponding *propagator*, which promotes these scalar Green's functions to elements of $\mathcal{A}_{pax} \boxtimes \mathcal{A}_{pax}$ and $\mathcal{A}_{ax} \boxtimes \mathcal{A}_{ax}$ respectively:

$$P_{pax}(x - x', y - y') = -\frac{|y - y'|}{2} \delta(x - x') (dx \otimes dx') e_a \otimes e_a \quad (2.8)$$

$$P_{ax}(x - x'; y, y') = \bar{G}_{ax}(y, y') \delta(x - x') (dx \otimes dx') e_a \otimes e_a \quad (2.9)$$

These propagators are integral kernels of Green's operators (with respect to the inner product pairing on \mathfrak{g} -valued forms) for $-\partial_y^2$ acting on \mathcal{A}_{pax} and \mathcal{A}_{ax} , respectively. Here the e_a , $a = 1, \dots, \dim G$ are an orthonormal basis for $\mathfrak{g} \cong \mathfrak{g}^*$ and we sum over repeated indices (so $e_a \otimes e_a$ denotes the identity tensor in $\text{Sym}^2(\mathfrak{g})$).

Evaluating Wilson loop expectations involves two procedures: (i) express the Wilson loop observable $W_{f,\gamma}(A)$ as a power series functional in A ; (ii) insert propagators, each weighted with λ , into all available slots using the Wick rule. We call each such insertion a *Wick contraction*.

For (i), it suffices to assume $f = \text{tr} \rho$, where $\rho : G \rightarrow \text{End}(V)$ is an irreducible unitary representation of G (since characters form an orthonormal basis in the space of class functions on G). As a consequence, we may as well assume \mathfrak{g} is embedded inside $\text{End}(V)$. In this way, we can assume that both f and the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} are multiples of trace on $\text{End}(V)$ restricted to G and \mathfrak{g} , respectively. With these assumptions, we can represent $\text{hol}_\gamma(A)$ as a path ordered exponential that is a power series element in $\text{End}(V)$, and then apply f :

$$W_{f,\gamma}(A) = f(\text{hol}_\gamma(A)) \quad (2.10)$$

$$= f(1) + \sum_{n=1}^{\infty} (-1)^n \int_{1 \geq t_n \geq \dots \geq t_1 \geq 0} f\left((\gamma^* A)(t_n) \cdots (\gamma^* A)(t_1)\right). \quad (2.11)$$

For (ii), when we have a propagator of the general form

$$P = G_{\mu\nu}(u; u') (dx^\mu \otimes dx^\nu) e_a \otimes e_a,$$

where u, u' are points of \mathbb{R}^2 and $\mu, \nu = 1, 2$, the sum over Wick contractions of (2.11) yields

$$\langle W_{f,\gamma} \rangle_P = \dim V + \sum_{n=1}^{\infty} \frac{\lambda^n}{2^n n!} \sum_{\sigma \in S_{2n}} (\text{Lie})_\sigma (An)_\sigma \quad (2.12)$$

where to each permutation $\sigma \in S_{2n}$ we have the corresponding *analytic factor*

$$(An)_\sigma = \int_{1 > t_{2n} > \dots > t_1 > 0} \left[G_{\mu_{\sigma(2n)} \mu_{\sigma(2n-1)}} \left(\gamma(t_{\sigma(2n)}); \gamma(t_{\sigma(2n-1)}) \right) \cdots \right. \\ \left. G_{\mu_{\sigma(2)} \mu_{\sigma(1)}} \left(\gamma(t_{\sigma(2)}); \gamma(t_{\sigma(1)}) \right) \right] \times \prod_{i=1}^{2n} \dot{\gamma}^{\mu_i}(t_{\sigma(i)}) dt^i \quad (2.13)$$

and *Lie factor*

$$(\text{Lie})_\sigma = \delta_{a_{\sigma(2n)} a_{\sigma(2n-1)}} \cdots \delta_{a_{\sigma(2)} a_{\sigma(1)}} \text{tr}(e_{a_{2n}} e_{a_{2n-1}} \cdots e_{a_2} e_{a_1}). \quad (2.14)$$

As mentioned in the introduction, (2.12) is a provisional definition, since in (2.13) the Green's functions G are singular. In our case, for P given by P_{ax} and P_{pax} , the integrals

(2.13) are finite since the singularities in G are tame, but for P_{ax} , we get expressions that depend on how we approximate the domain of integration. To avoid singularities along the diagonal of G , we took all inequalities $t_{i+1} > t_i$ in (2.13) to be strict.

In computing Wilson loops, we have to specify the class the loops γ which we are considering. In axial gauge, our (arbitrarily) chosen y -direction is a distinguished direction. Thus, it is natural to decompose curves into those which are vertical segments and those which are horizontal curves:

Definition 2.1. A *curve* γ is a piecewise C^1 -embedding of an interval I (by default $[0, 1]$) into \mathbb{R}^2 . Since most of our constructions will not depend on how γ is parametrized, we often conflate γ with its image in \mathbb{R}^2 . A curve $\gamma : [x_-, x_+] \rightarrow \mathbb{R}^2$ is *horizontal* if it (or its parametrization-reversed curve) is given by $\gamma(t) = (t, \bar{\gamma}(t))$ with $\bar{\gamma}(t)$ continuously differentiable. We say that a horizontal curve is *right-moving* or *left-moving* if the x -coordinate of γ is increasing or decreasing, respectively.

Definition 2.2. A curve γ in the plane is called *admissible* if it can be written as a concatenation $\gamma = \gamma_m \cdots \gamma_1$ with each γ_i a horizontal curve or a vertical segment (we concatenate from right to left). We call such a decomposition an *admissible curve decomposition*. We write

$$\gamma \equiv \gamma_m \cdots \gamma_1 \tag{2.15}$$

if γ is the concatenation of the horizontal curves γ_i up to vertical segments (i.e. we simply omit the vertical segments in an admissible curve decomposition for γ). We say that (2.15) is a *horizontal curve decomposition* for γ .

In axial gauge, parallel transport along a vertical segment is trivial since the connection has no dy component. Thus, to determine the parallel transport along an admissible curve γ , it suffices to know its horizontal curve decomposition.

Any curve γ can be approximated in C^0 by an admissible curve. Indeed, a piecewise linear approximation $\gamma^\#$ of γ will be admissible. In fact, such an approximation can also be chosen so as to make

$$\int_0^1 |\dot{\gamma}(t) - \dot{\gamma}^\#(t)| dt$$

arbitrarily small, i.e., $\gamma^\#$ is “piecewise C^1 -close” to γ . It follows that to prove our main theorem for arbitrary closed curves, it suffices to establish it for curves which are admissible (or even piecewise linear).

2.1. Generalized White-Noise. In what follows, we put in quotation marks terms borrowed from probability theory since the intuition they provide is apt. We think of the (x -component of) our axial-gauge connection $A^a(x, y)$ as being a Lie-algebra valued “Gaussian free field” distributed according to

$$\left\langle A^a(x, y) A^b(x', y') \right\rangle = \delta^{ab} \lambda G(x, y; y', y'),$$

with G the corresponding Green’s function. Since our Wilson loop operators are obtained by performing parallel transport with respect to the $A^a(x, y)$, our basic “random variables” should be obtained by integration of the $A^a(x, y)$ against curves.

Definition 2.3. To each horizontal curve $\gamma : [x_-, x_+] \rightarrow \mathbb{R}^2$ and $a = 1, \dots, \dim(G)$, we define “white-noise” maps $M^{\gamma, a}$ as follows. For every $f \in L^2(\mathbb{R})$, the $M^{\gamma, a}(f)$ form a collection a “Gaussian random variables”, whose covariance is given by

$$\mathbb{E}\left(M^{\gamma_1, a_1}(f_1)M^{\gamma_2, a_2}(f_2)\right) = \pm \delta^{a_1 a_2} \lambda \int_{x_-^2}^{x_+^2} \int_{x_-^1}^{x_+^1} f_1(x_1) f_2(x_2) G(x_1, \bar{\gamma}_1(x_1); x_2, \bar{\gamma}_2(x_2)) dx_2 dx_1 \quad (2.16)$$

where the \pm is determined by whether the γ_i move in the same direction or opposite direction, respectively. We set

$$M^\gamma = M^{\gamma, a} e_a.$$

to obtain \mathfrak{g} -valued “white-noise”. For notational convenience, given an interval I , we define

$$M^\gamma(I) = M^\gamma(\mathbf{1}_I)$$

where $\mathbf{1}_I$ is the indicator function of I .

Remark 2.4. For the usual (one-dimensional, \mathbb{R} -valued) white-noise construction, we replace \mathbb{R}^2 with \mathbb{R} , let $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ be the identity map, and let $G(x, x') = \delta(x - x')$ in the above. This yields $\langle M^\gamma(f_1), M^\gamma(f_2) \rangle = \langle f_1, f_2 \rangle_{L^2}$. The process $x \mapsto M^\gamma([0, x])$ is distributed as Brownian motion.

For the partial axial-gauge and complete axial-gauge, we have

$$G(x, y; x', y') = \bar{G}(y, y') \delta(x - x') \quad (2.17)$$

with $\bar{G}(y, y')$ the appropriate function of y , and so (2.16) becomes

$$\mathbb{E}\left(M^{\gamma_1, a}(I_1)M^{\gamma_2, b}(I_2)\right) = \pm \delta^{ab} \lambda \int_{I_1 \cap I_2} \bar{G}(\bar{\gamma}_1(x), \bar{\gamma}_2(x)) dx. \quad (2.18)$$

For $\bar{G}(y, y')$ given by complete axial-gauge, the induced integral operator is positive definite since $\bar{G}_{ax}(y, y')$ is the covariance for two independent Brownian motions (moving to the left and right of the origin). However, for $\bar{G}_{pax}(y, y') = -\frac{1}{2}|y - y'|$ given by partial axial-gauge, we obtain an indefinite operator. Indeed, it is easy to arrange for

$$\int f(y) dy \int \left(-\frac{1}{2}|y - y'|\right) f(y') dy' \quad (2.19)$$

to be negative since $\bar{G}_{pax}(y, y') \leq 0$, but if we let f be the sum of two widely spaced bump functions of opposite sign, (2.19) will be positive.

In this manner, we are led to repeat stochastic analysis with white-noise maps having indefinite covariance, since this is what arises when considering parallel transport operators in partial axial-gauge. However, it turns out that our analysis becomes one of a rather general nature, not just one tied to the specifics of partial axial-gauge. Thus, we develop an abstract framework in the next section that works with general white-noise maps having indefinite covariance. Specializing this abstract framework to the particular white-noise map in Definition 2.3, we prove our main results in Section 4.

3. ALGEBRAIC STOCHASTIC CALCULUS

Fix a compact Lie algebra $\mathfrak{g} \subset \text{End}(V)$ and a positive integer m . Consider an algebra of “random variables” $M^{\alpha,a}(I)$ indexed by closed intervals I and $\alpha \in \{0, 1, \dots, m\}$, $a \in \{1, 2, \dots, \dim \mathfrak{g}\}$. This algebra is such that if I and J have disjoint interior, then

$$M^{\alpha,a}(I \cup J) = M^{\alpha,a}(I) + M^{\alpha,a}(J).$$

Definition 3.1. Let $C(x) = C^{\alpha\beta}(x)$ be a continuous, symmetric matrix, $0 \leq \alpha, \beta \leq m$ and λ a formal variable. The *expectation* operator \mathbb{E} is the linear functional defined on the algebra generated by the $M^{\alpha,a}(I)$ given by

$$\mathbb{E}(M^{\alpha,a}(I)M^{\beta,b}(J)) = \delta^{ab}\lambda \int_{I \cap J} C^{\alpha\beta}(x)dx, \quad (3.1)$$

and which extends to all other monomials by the Wick rule:

$$\mathbb{E}(X_1 \cdots X_n) = \begin{cases} \frac{1}{2^{n/2}(n/2)!} \sum_{\sigma \in S_n} \mathbb{E}(X_{\sigma(1)}X_{\sigma(2)}) \cdots \mathbb{E}(X_{\sigma(n-1)}X_{\sigma(n)}) & n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases} \quad (3.2)$$

The operator \mathbb{E} extends to the algebra generated by $M^\alpha(I) := M^{\alpha,a}(I)e_a$ via linearity over $\text{End}(V)$.

We are of course interested in the case $C^{\alpha\beta}(x) = \pm \frac{1}{2}|\bar{\gamma}_\alpha(x) - \bar{\gamma}_\beta(x)|$ relevant to partial axial-gauge, where γ_α are a collection of horizontal curves. We ultimately will interpret λ as a positive real number, but it is convenient to consider it as a formal parameter in the present general setting.

The intervals we consider will always arise from subdivision of a fixed finite interval. Without loss of generality, we suppose this interval is $[0, L]$ for some $L > 0$. We let $N > 1$ be our subdivision (i.e. discretization) parameter, with $N \rightarrow \infty$ representing a continuum limit. Our ultimate aim in this section is to prove Lemmas 3.11 and 3.12. These both strongly rely on the discretization procedure being well-crafted and is the cause for the level of detail in our constructions.

Let $\Delta x = L/N$ and

$$\mathcal{I}_N = \{I_i = [i\Delta x, (i+1)\Delta x] : 0 \leq i \leq N-1\}$$

be the set of intervals obtained from uniform subdivision of $[0, L]$ into N intervals. These intervals will be used to define discretized integrals (Riemann, Ito, and Stratonovich). Because Stratonovich integrals use the midpoint rule, we also need to consider $\Delta x/2$ -translates of these intervals intersected with $[0, L]$. We thus obtain the set of intervals

$$\tilde{\mathcal{I}}_N = \{J_i = [i\Delta x/2, (i+2)\Delta x/2] : 0 \leq i \leq 2N-2\} \cup \{J_{-1} = [0, \Delta x/2]\}$$

Let

$$\Lambda_N = \{i\Delta x : 0 \leq i \leq N\}.$$

Given a lattice interval $[x_-, x_+]$, with $x_- = i_- \Delta x$ and $x_+ = i_+ \Delta x$, we can consider

$$\mathcal{I}_N[x_-, x_+] = \{I_k : i_- \leq k \leq i_+ - 1\},$$

the set of intervals I_j in \mathcal{I}_N that are contained in $[x_-, x_+]$. For general $0 \leq x_-, x_+ \leq L$, we can define $\mathcal{I}_N[x_-, x_+]$ to be $\mathcal{I}_N[\hat{x}_-, \hat{x}_+]$ with \hat{x}_\pm the point in Λ_N nearest to and less than or equal to x_\pm .

Given a function f defined on Λ_N , we will denote its evaluation at $x \in \Lambda_N$ by $f(x)$ or f_x .

Definition 3.2. Let f be an $(\text{End}(V))$ -valued function defined on Λ_N and fix α . Then given any interval $[x, x'] \subseteq [0, L]$, we can define the following sums:

(i) *Ito sum*:

$$\sum_{I \in \mathcal{I}_N[x, x']} M^\alpha(I) f(x_I^-)$$

where x_I^- is the left endpoint of I .

(ii) *Stratonovich sum*:

$$\sum_{I \in \mathcal{I}_N[x, x']} M^\alpha(I) f(\bar{x}_I)$$

where \bar{x}_I is the midpoint of I .

(iii) *Riemann sum*:

$$\sum_{I \in \mathcal{I}_N[x, x']} g(\tilde{x}_I^*) |I| f(x_I^*)$$

where \tilde{x}_I^* and x_I^* are arbitrary points of I and $I \cap \Lambda_N$, respectively, and g is a continuous function on $[x, x']$.

In the above sums, right instead of left multiplication by the $M^\alpha(I)$ can be considered as well.

The Ito and Stratonovich sums are elements of the algebra generated by the M^α and not ordinary numbers. That being the case, the usual tools for estimating sums (e.g. norm-bounds) do not apply. While we can apply \mathbb{E} to Ito and Stratonovich sums, the resulting numerical sums cannot be estimated using basic tools such as applying the Cauchy-Schwarz inequality to the pairing (3.1), since the pairing is not necessarily positive definite. Thus, we have to introduce a bit of terminology in order to organize our estimates.

Let \mathcal{H}_N denote the vector space with basis elements indexed by the elements of $\tilde{\mathcal{I}}_N$. Then we can regard the M^α as maps

$$M^\alpha : \mathcal{H}_N \rightarrow \mathfrak{g} \subset \text{End}(V).$$

Thus, polynomials in the M^α become elements of $\text{Sym}(\mathcal{H}_N^*) \otimes \text{End}(V)$. These all belong to the larger space

$$\mathcal{F}_N := \widehat{\text{Sym}}(\mathcal{H}_N^*) \otimes \text{End}(V)$$

of power series elements in \mathcal{H}_N^* with values in $\text{End}(V)$. Given $f \in \widehat{\text{Sym}}(\mathcal{H}_N^*) \otimes \text{End}(V)$, write $f^{(i)}$ to denote the degree i part of f (with respect to the natural grading of $\widehat{\text{Sym}}(\mathcal{H}_N^*)$). We think of elements of \mathcal{F}_N as “functions”.

We will need to consider inverses of elements in \mathcal{F}_N . An element $h \in \widehat{\text{Sym}}(\mathcal{H}_N^*) \otimes \text{End}(V)$ has a multiplicative inverse if and only if its constant part is an invertible element of $\text{End}(V)$.

A priori, an element f of $\widehat{\text{Sym}}(\mathcal{H}_N) \otimes \text{End}(V)$ only has a well-defined finite expectation for each fixed polynomial degree: the resulting series $\sum_i \mathbb{E}(f^{(i)})$ may not converge in $\text{End}(V)$. However, by regarding λ as a formal parameter, we see that $\sum_i \mathbb{E}(f^{(i)})$ makes sense as a formal power series in λ . This allows us to extend the definition of \mathbb{E} on polynomials to formal power series.

Given an invertible element $h \in \widehat{\text{Sym}}(\mathcal{H}_N) \otimes \text{End}(V)$, define the corresponding right-adjoint action

$$\begin{aligned} \text{ad } h : \widehat{\text{Sym}}(\mathcal{H}_N) \otimes \text{End}(V) &\rightarrow \widehat{\text{Sym}}(\mathcal{H}_N) \otimes \text{End}(V) \\ X &\mapsto h^{-1} X h. \end{aligned}$$

We have $\text{tr} X = \text{tr}(\text{ad}(h)X)$ by cyclicity of trace and since $\widehat{\text{Sym}}(\mathcal{H}_N^*)$ is a commutative algebra.

Let $\mathcal{X}_N = \{M^\alpha(I) : 0 \leq \alpha \leq m, I \in \tilde{\mathcal{I}}_N\}$. We think of the \mathcal{X}_N as infinitesimals, i.e. as differentials, since they are formed out of intervals that are of size $O(N^{-1})$.

Definition 3.3. An *admissible sum* is an iterated sum (any combination of Ito, Stratonovich, or Riemann), with each sum being determined by an N -independent interval. An *admissible polynomial* is any (N -independent) linear combination of products of admissible sums. We regard such objects as a sequence of elements $f = (f_N)$, $f_N \in \mathcal{F}_N$.

An *admissible function* $f = (f_N)$ is one such that each $f_N^{(i)} \in \mathcal{F}_N$ is a linear combination of products of admissible polynomials and elements of \mathcal{X}_N . Since each \mathcal{F}_N is an algebra, admissible functions form an algebra under component-wise multiplication.

We can regard the $M^\alpha(I)$ as admissible functions in the sense that I is a variable element of $\tilde{\mathcal{I}}_N$ as N varies.

Definition 3.4. Let f be an admissible function. We write

$$f = O(N^{-k}) \tag{3.3}$$

if given any admissible functions g_1 and g_2 and $i \geq 0$,

$$\left| \mathbb{E} \left((g_1 f g_2)_N^{(i)} \right) \right| \leq C N^{-k} \tag{3.4}$$

for all N . Here C is independent of N .

We write

$$f = \underline{O}(N^{-k}) \tag{3.5}$$

if $f = O(N^{-k})$ and for every $g = O(N^{-k'})$, we have $fg = O(N^{-(k+k')})$.

We write $f = O(g_1^{n_1} \cdots g_k^{n_k})$ if f contains the g_i as a multiplicative factors, each with multiplicity at least n_i . We also write $f = O(I^n)$ with $I \in \tilde{\mathcal{I}}_N$ if $f = O(M^{\alpha_1}(I_1) \cdots M^{\alpha_n}(I_n))$ with I_i approximately I in the sense that $|I_i| - |I| = O(N^{-1})$ and $\text{dist}(I, I_i) = O(N^{-1})$.

In particular, if $f = O(N^{-k})$ with $k > 0$, then

$$\lim_{N \rightarrow \infty} \mathbb{E}(f) = 0.$$

One subtlety with the $O(N^{-k})$ notation is that it is not additive with respect to multiplication of differentials:

$$\begin{aligned} M^\alpha(I) &= O(N^{-1}) \\ M^\alpha(I)M^\beta(J) &= O(N^{-1}), \quad I, J \in \tilde{\mathcal{I}}_N. \end{aligned}$$

This is analogous to the case of stochastic differentials, leading to peculiar phenomenon such as that which occurs in Ito's formula. This is the reason we introduce the \underline{O} notation. However, if I and J are disjoint elements of $\tilde{\mathcal{I}}_N$, then

$$M^\alpha(I)M^\beta(J) = O(N^{-2}), \quad I \cap J = \emptyset.$$

Admissible sums are essentially multiple integrals discretized using the lattice Λ_N or Λ_{2N} . To keep our already involved notation to a minimum, we focus on the kinds of integrals and sums relevant to the problem at hand.

Define variables $M_x^{\alpha,a}$ satisfying

$$\mathbb{E}(M_x^{\alpha,a} M_{x'}^{\beta,b}) = \delta^{ab} \delta(x - x') C^{\alpha\beta}(x) \quad (3.6)$$

Thus,

$$\int_I M_x^\alpha dx = M^\alpha(I). \quad (3.7)$$

The true meaning of these definitions is that one obtains a well-defined expectation of integrals of the form

$$I(\vec{\alpha}) := \int_{L > x_n > \dots > x_1 > 0} M_{x_n}^{\alpha_n} \dots M_{x_1}^{\alpha_1} dx_n \dots dx_1, \quad \vec{\alpha} = (\alpha_1, \dots, \alpha_n). \quad (3.8)$$

by use of (3.6) and the Wick rule. From a more rigorous standpoint, one can regard (3.8) as a limit of iterated Ito sums (see Lemma 3.5), each of which is determined by approximating the simplex $\{L > x_n > \dots > x_1 > 0\}$ with open n -cubes and then applying the formula (3.7). Indeed, consider

$$I_N(\vec{\alpha}) = \int_{L > x_n >_N \dots >_N x_1 > 0} M_{x_n}^{\alpha_n} \dots M_{x_1}^{\alpha_1} \quad (3.9)$$

where³

$$x >_N y \Leftrightarrow x \in I_i, y \in I_j, i > j \quad I_i, I_j \in \mathcal{I}_N. \quad (3.10)$$

Then the domain of I_N approximates the domain of I , and $L > x_n >_N \dots >_N x_1 > 0$ can be written as union of cubes $I_{i_n} \times \dots \times I_{i_1}$. Applying (3.7) we can then rewrite $I_N(\vec{\alpha})$ as the iterated Ito sum

$$S(\vec{\alpha}) := \sum_{\substack{i_n > \dots > i_1 \\ I_{i_k} \in \mathcal{I}_N}} M^{\alpha_n}(I_{i_n}) \dots M^{\alpha_1}(I_{i_1}). \quad (3.11)$$

Note that it was crucial that the inequalities in (3.8) were strict in order for it to be approximated by an Ito sum (3.11).

³Technically speaking, the condition $x >_N y$ allows for $x = y$ at the points of Λ_N where adjacent intervals meet. But since Λ_N is a discrete set, this set of coincidence points is immaterial, i.e. does not contribute to (3.8). While this involves some heuristic reasoning with the formal expression (3.8), this can be seen at the discretized level by noting that the closed intervals I_j can be replaced by their interior.

Lemma 3.5. *The limit*

$$I(\vec{\alpha}) := \lim_{N \rightarrow \infty} I_N(\alpha)$$

exists in the sense that the $I_N(\alpha)$ satisfy

$$I_N(\alpha) - I_M(\alpha) = \underline{O}(N^{-1}), \quad N \leq M.$$

In particular, the expectation of I times any admissible function is well-defined. Moreover, given $\vec{\alpha}^{(i)}$ for $1 \leq i \leq m$, then

$$I(\vec{\alpha}^{(1)}) \cdots I(\vec{\alpha}^{(m)}) - I_N(\vec{\alpha}^{(1)}) \cdots I_N(\vec{\alpha}^{(m)}) = \underline{O}(N^{-1}). \quad (3.12)$$

PROOF. It suffices to prove the case $m = 1$, since

$$\begin{aligned} I(\vec{\alpha}^{(1)}) \cdots I(\vec{\alpha}^{(m)}) - I_N(\vec{\alpha}^{(1)}) \cdots I_N(\vec{\alpha}^{(m)}) &= \left(I(\vec{\alpha}^{(1)}) - I_N(\vec{\alpha}^{(1)}) \right) I(\vec{\alpha}^{(2)}) \cdots I(\vec{\alpha}^{(m)}) + \cdots \\ &\quad + I_N(\vec{\alpha}^{(1)}) \cdots I_N(\vec{\alpha}^{(m-1)}) \left(I(\vec{\alpha}^{(m)}) - I_N(\vec{\alpha}^{(m)}) \right). \end{aligned}$$

To that end, we first analyze the difference between the domain of integration for $I(\vec{\alpha})$ and that of $I_N(\vec{\alpha})$. It is given by a union of the forbidden regions

$$D_i = \{L > x_n > \cdots > x_1 > 0 : x_{i+1} \not\prec_N x_i\}.$$

While the path-ordered simplex $\{L > x_n > \cdots > x_1 > 0\}$ has fixed volume, the regions D_i have volume $O(N^{-1})$, since they have width $O(N^{-1})$ in the $x_i - x_{i+1}$ direction. It is thus enough to replace the left-hand side of (3.12) for $m = 1$, with the integrals

$$\sum_{i=1}^n \int_{D_i} M_{x_n}^{\alpha_n} \cdots M_{x_1}^{\alpha_1} dx_n \cdots dx_1. \quad (3.13)$$

(The D_i are not disjoint, but their overlaps are “codimension two”, i.e., have volume $O(N^{-2})$ and so will be of lower order.) Technically, we should be considering $I_N - I_M$ instead of (3.13), but the former’s domain of integration will be covered by the D_i , so the same proof given below will apply.

If we multiply (3.13) by any admissible function, the resulting expectation is $O(N^{-1})$ because the volume of the D_i are $O(N^{-1})$. Suppose we multiply (3.13) by an admissible function $f = O(N^{-k})$, so that it has at least k differentials $M^{\beta_i}(I_j)$ occurring with $|I_j| = O(N^{-1})$. The only way the order of f times (3.13) can drop is if in the sum over Wick contractions occurring between a fixed $M^{\beta_i}(I_j)$ of f and the terms of (3.13), we obtain a number that is of order N^{-1} instead of N^{-2} (i.e. we have two factors of order $O(N^{-1})$ yielding a term of order $O(N^{-1})$). However, it is easy to see that the set of points in D_i which possesses at least one coordinate belonging to a fixed I_j (this is the set where the desired Wick contractions can occur) has volume of order $O(N^{-2})$. Indeed, one must constrain both a coordinate function and the $x_i - x_{i+1}$ separation to be $O(N^{-1})$. Thus, the order of f times (3.13) cannot drop. \square

Next, we define discretized versions of the path-ordered exponential. Recall that such a path-ordered exponential represents the solution of an ordinary differential equation, which has the geometric interpretation of parallel transport in the setting of gauge theory. In the (algebraic) stochastic setting, the iterated integrals that appear in the series expansion of parallel transport can be recast as Ito or Stratonovich integrals (sums).

Definition 3.6. Fix α , an initial point x_- , and a terminal point x . Define *Ito parallel transport* from x_- to x via

$$P_{N,x_- \rightarrow x}^{M^\alpha} = \begin{cases} 1 + \sum_{n=1}^{\infty} (-1)^n \sum_{\substack{i_n > \dots > i_1 \\ I_{i_k} \subseteq \tilde{\mathcal{I}}_N[x_-, x]}} M^\alpha(I_{i_n}) \cdots M^\alpha(I_{i_1}) & x \geq x_- \\ 1 + \sum_{n=1}^{\infty} \sum_{\substack{i_n < \dots < i_1 \\ I_{i_k} \in \tilde{\mathcal{I}}_N[x, x_-]}} M^\alpha(I_{i_n}) \cdots M^\alpha(I_{i_1}) & x \leq x_- \end{cases}$$

We call these two cases *right-moving* and *left-moving* Ito parallel transport, respectively. The above sums are iterated Ito sums since consecutive I_i overlap at their common endpoint.

Define the relations

$$i \succeq j \Leftrightarrow j = i, i-2, i-4, \dots \quad (3.14)$$

$$i \succ j \Leftrightarrow j = i-1, i-3, i-5, \dots \quad (3.15)$$

and similarly with \preceq and \prec .

Definition 3.7. For $x \geq x_-$, let i_+ be the maximum i such that J_i belongs to $\tilde{\mathcal{I}}_N[x_-, x]$. Likewise, for $x \leq x_-$, let i_- be the minimum i such that J_i belongs to $\tilde{\mathcal{I}}_N[x, x_-]$. Then define *Stratonovich parallel transport* from x_- to x_+ via

$$\tilde{P}_{N,x_- \rightarrow x}^{M^\alpha} = \begin{cases} 1 + \sum_{n=1}^{\infty} (-1)^n \sum_{\substack{i_+ \succeq i_n \succ \dots \succ i_1 \\ J_{i_k} \subseteq \tilde{\mathcal{I}}_N[x_-, x]}} M^\alpha(J_{i_n}) \cdots M^\alpha(J_{i_1}) & x \geq x_- \\ 1 + \sum_{n=1}^{\infty} \sum_{\substack{i_- \preceq i_n \prec \dots \prec i_1 \\ J_{i_k} \in \tilde{\mathcal{I}}_N[x, x_-]}} M^\alpha(J_{i_n}) \cdots M^\alpha(J_{i_1}) & x \leq x_- \end{cases}$$

We call these two cases *right-moving* and *left-moving* Stratonovich parallel transport, respectively. The above sums are iterated Stratonovich sums since consecutive J_i overlap halfway.

In stochastic calculus, one can convert Stratonovich integrals to Ito integrals. The same idea allows us to convert from Stratonovich sums to Ito sums up to an error of order $\underline{O}(N^{-1})$. Define

$$i \succ \succ j \Leftrightarrow j = i-2, i-4, \dots$$

Lemma 3.8. (*Stratonovich to Ito conversion*) *We have*

$$\begin{aligned}
& \sum_{\substack{i_+ \succ i_n \succ \dots \succ i_1 \\ J_{i_k} \subseteq \tilde{\mathcal{I}}_N[x^-, x]}} M^{\alpha_n}(J_{i_n}) \cdots M^{\alpha_1}(J_{i_1}) \\
&= \sum_{\substack{i_+ \succ i_n \succ \dots \succ i_{n-1} \succ \dots \succ i_1 \\ J_{i_k} \subseteq \tilde{\mathcal{I}}_N[x^-, x]}} M^{\alpha_n}(J_{i_n}) \cdots M^{\alpha_1}(J_{i_1}) \\
&+ \sum_{\substack{i_+ \succ i_n \succ \dots \succ i_{n-2} \succ \dots \succ i_1 \\ i_{n-1} = i_n - 1 \\ J_{i_k} \subseteq \tilde{\mathcal{I}}_N[x^-, x]}} \mathbb{E} \left(M^{\alpha_n}(J_{i_n}) M^{\alpha_{n-1}}(J_{i_{n-1}}) \right) M^{\alpha_{n-2}}(J_{i_{n-2}}) \cdots M^{\alpha_1}(J_{i_1}) \\
&+ \underline{O}(N^{-1})
\end{aligned}$$

and similarly when iterating sums from left to right.

In other words, in going from the left-hand side to the right-hand side in the above equation, the Stratonovich sum over the i_n index was converted an Ito sum at the expense of an expectation of the i_n and i_{n-1} terms (which is nonzero only if $i_{n-1} = i_n - 1$, i.e., if $J_{i_{n-1}}$ and J_{i_n} overlap) and an error of order $\underline{O}(N^{-1})$. Iterating this procedure we can convert all Stratonovich sums into Ito–Riemann sums up to $\underline{O}(N^{-1})$.

PROOF. We need to show that

$$\sum_{\substack{i_+ \succ i_n \\ J_{i_n} \in \mathcal{I}_N[x^-, x]}} \left[M^{\alpha_n}(J_{i_n}) M^{\alpha_{n-1}}(J_{i_{n-1}}) - \mathbb{E} \left(M^{\alpha_n}(J_{i_n}) M^{\alpha_{n-1}}(J_{i_{n-1}}) \right) \right] \quad (3.16)$$

is $\underline{O}(N^{-1})$. First, each term of the sum of (3.16) is $O(N^{-2})$ since it is quadratic in the $M^\alpha(J)$ and has zero expectation. On the other hand, we have a sum over $O(N)$ many terms. The same analysis in the proof of Lemma 3.5 shows that we obtain a result that is $\underline{O}(N^{-1})$, since the order of (3.16) cannot drop when multiplying by a differential (the terms of (3.16) have supports that are “sparse”, so that if we wish to consider those that have support on an interval of length $O(N^{-1})$, we pick up only finitely many terms). \square

Given an interval $J = [x, x + \Delta x] \in \tilde{\mathcal{I}}_N$, write

$$\begin{aligned}
J^- &= [x, x + \Delta x/2] \\
J^+ &= [x + \Delta x/2, x + \Delta x].
\end{aligned}$$

to denote the subintervals obtained from division of J along its midpoint. For the special interval $J_{-1} \in \tilde{\mathcal{I}}_N$ of length $\Delta x/2$, define

$$\begin{aligned}
J_{-1}^- &= \{0\} \\
J_{-1}^+ &= J_{-1}.
\end{aligned}$$

These intervals belong to \mathcal{I}_{2N} . They satisfy

$$J_i^+ = [(i+1)\Delta x/2, (i+2)\Delta x/2], \quad -1 \leq i \leq 2N-2 \quad (3.17)$$

$$J_i^+ = J_{i+1}^- \quad (3.18)$$

so that the J_i^+ partition $[0, L]$ and intersect only at common endpoints.

Write $(f \circ M)(J)$ to mean $f(\bar{x})M(J)$ where \bar{x} is the midpoint of J (this assumes that for f defined on a lattice, \bar{x} is a corresponding lattice point). Define

$$(f \delta M)(J) = (f \circ M)(J^-) + (f \circ M)(J^+) \quad (3.19)$$

From our M^α , $0 \leq \alpha \leq m$, we isolate $\alpha = 0$ and define

$$h_{N,x} = P_{N,0 \rightarrow x}^{M^0}, \quad x \in \Lambda_N. \quad (3.20)$$

We think of h_N as a gauge-transformation defined on the lattice site x . It satisfies a discretized Ito differential equation:

$$h_{N,x+\Delta x} - h_{N,x} = -M^0([x, x+\Delta x])h_{N,x}.$$

Lemma 3.9. *Let $J \in \tilde{I}_N$. For $x \in \Lambda_{4N}$, let $I_x = [x, x+\Delta x/4]$. We have*

$$\begin{aligned} \text{ad}(h_{4N,x+\Delta x/4})M^\alpha(J) &= \text{ad}(h_{4N,x})M^\alpha(J) + O([M^0(I_x), M^\alpha(J)]) + O(M^0(I_x)^2, M^\alpha(J)) \\ &= \text{ad}(h_{4N,x})M^\alpha(J) + O(N^{-2}). \end{aligned}$$

PROOF. For any increment Δx , we have

$$\begin{aligned} \left(\text{ad}(h_{4N,x+\Delta x}) - \text{ad}(h_{4N,x}) \right) X &= \left(h_{4N,x+\Delta x}^{-1} - h_{4N,x}^{-1} \right) X h_{4N,x} + h_{4N,x}^{-1} X \left(h_{4N,x+\Delta x} - h_{4N,x} \right) + \\ &\quad + \left(h_{4N,x+\Delta x}^{-1} - h_{4N,x}^{-1} \right) X (h_{4N,x+\Delta x} - h_{4N,x}) \end{aligned}$$

Now, when we do a single increment $\Delta x/4$ for $h_{4N,x}$ we get

$$\begin{aligned} h_{4N,x+\Delta x/4} &= (1 - M^0(I_x))h_{4N,x} \\ h_{4N,x+\Delta x/4}^{-1} &= h_{4N,x}^{-1}(1 - M^0(I_x))^{-1} \\ &= h_{4N,x}^{-1}(1 + M^0(I_x) + \dots). \end{aligned}$$

Hence,

$$\begin{aligned} \left(\text{ad}(h_{4N,x+\Delta x/4}) - \text{ad}(h_{4N,x}) \right) M^\alpha(J) &= \text{ad}(h_{4N,x})[M^0(I_x), M^\alpha(J)] \\ &\quad + O(M^0(I_x)^2, M^\alpha(J)). \end{aligned} \quad (3.21)$$

Now,

$$\begin{aligned} \mathbb{E}([M^0(J_x), M^\alpha(J)]) &= [e^a, e^b] \mathbb{E}(M^{0,a}(I_x)M^{\alpha,b}(J)) \\ &= 0. \end{aligned}$$

since the expectation is nonvanishing only for $a = b$, in which case $[e^a, e^a] = 0$. So the right-hand side of (3.21) is $O(N^{-2})$. \square

Lemma 3.10. *Let $J \in \tilde{\mathcal{I}}_N$. We have*

$$\text{ad}(h_{2N}) \circ M^\alpha(J) = \text{ad}(h_{4N}) \hat{\circ} M^\alpha(J) + O(N^{-2})$$

with the remainder $O(N^{-2})$ consisting of terms of the form $O(J^2)$ and $\underline{O}(N^{-1})O(J)$.

PROOF. For $J = [x, x + \Delta x]$, we have

$$\begin{aligned} \text{ad}(h_{4N}) \hat{\circ} M^\alpha(J) &= \text{ad}(h_{4N, x+\Delta x/4}) M^\alpha(J^-) + \text{ad}(h_{4N, x+3\Delta x/4}) M^\alpha(J^+) \\ \text{ad}(h_{2N}) \circ M^\alpha(J) &= \text{ad}(h_{2N, x+\Delta x/2}) M^\alpha(J^-) + \text{ad}(h_{2N, x+\Delta x/2}) M^\alpha(J^+) \\ &= \text{ad}(h_{4N, x+\Delta x/2}) M^\alpha(J^-) + \text{ad}(h_{4N, x+\Delta x/2}) M^\alpha(J^+) + \underline{O}(N^{-1})O(J) \end{aligned}$$

where we used Lemma 3.5 in the last line. Now apply the previous lemma. \square

Let x_\pm^α be points in $[0, L]$, $1 \leq \alpha \leq m$, satisfying the matching conditions

$$x_+^1 = x_-^2, \quad x_+^2 = x_-^3, \quad \dots, \quad x_+^m = x_-^1. \quad (3.22)$$

Define the function g_N^α on Λ_N using either of the following choices

$$g_{N,x}^\alpha = \begin{cases} P_{N,x_-^\alpha \rightarrow x}^{M^\alpha} & \text{only if } C^{\alpha\alpha} \equiv 0 \\ \tilde{P}_{N,x_-^\alpha \rightarrow x}^{M^\alpha} & \text{general } C^{\alpha\beta}. \end{cases} \quad (3.23)$$

We want to compute

$$\lim_{N \rightarrow \infty} \mathbb{E} \left(\text{tr}(g_{N,x_+^m}^m \cdots g_{N,x_+^1}^1) \right). \quad (3.24)$$

For $C^{\alpha\beta}$ pertaining to partial or complete axial-gauge, (3.24) is precisely our corresponding Wilson loop expectation. We can rewrite the trace occurring in (3.24) as

$$\text{tr} \left(h_{N,x_+^m}^{-1} g_{N,x_+^m}^m h_{N,x_-^m} h_{N,x_-^{m-1}}^{-1} g_{N,x_-^{m-1}}^{m-1} h_{N,x_+^{m-1}} \cdots h_{N,x_+^1}^{-1} g_{N,x_+^1}^1 h_{N,x_-^1} \right).$$

due to the matching condition (3.22).

So consider

$$\hat{g}_{N,x}^\alpha = h_{2N,x}^{-1} g_{2N,x}^\alpha h_{2N,x_-^\alpha}, \quad x \in \Lambda_{2N}. \quad (3.25)$$

The use of $2N$ on the right-hand side of (3.25) is to facilitate analysis at midpoints needed when working with Stratonovich sums. If we regard $g_{2N,x}^\alpha$ as parallel transport induced by M^α , we are to regard $\hat{g}_{N,x}^\alpha$ as parallel transport induced by the gauge-transform of M^α by h_{2N} .

Let

$$\tilde{M}^\alpha = M^\alpha - \sigma_\alpha M^0, \quad \alpha = 1, \dots, m. \quad (3.26)$$

where $\sigma_\alpha = \pm$ is determined by whether $g_{N,x}^\alpha$ is right-moving or left-moving, respectively (here x is fixed). These variables satisfy

$$\mathbb{E}(\tilde{M}^\alpha(I), \tilde{M}^\beta(J)) = \int_{I \cap J} \tilde{C}^{\alpha\beta}(x) dx \quad (3.27)$$

where

$$\tilde{C}^{\alpha\beta}(x) = C^{\alpha\beta}(x) - \sigma_\alpha C^{\alpha 0}(x) - \sigma_\beta C^{0\beta}(x) + \sigma_\alpha \sigma_\beta C^{00}(x). \quad (3.28)$$

Lemma 3.11. (*Change of gauge*) Suppose $g_{2N,x}^\alpha$ is right-moving. Then

$$\hat{g}_{N,x}^\alpha = 1 + \sum_{n=1}^{\infty} (-1)^n \sum_{\substack{i_+ \succ i_n \succ \dots \succ i_1 \\ J_{i_k} \subseteq \mathcal{I}_N[x_-^\alpha, x]}} (\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)(J_{i_n}) \cdots (\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)(J_{i_1}) + O(N^{-1}) \quad (3.29)$$

The analogous result holds for $g_{2N,x}^\alpha$ left-moving.

Notice that the sum in (3.29) is an iterated Stratonovich sum. Because $\mathbb{E}(\tilde{M}^\alpha(I), \tilde{M}^\alpha(J)) \neq 0$ for overlapping intervals, this remains distinct from an iterated Ito sum as $N \rightarrow \infty$.

PROOF. We consider the case when g_N^α is defined by the first case of (3.23), with the second case following similar lines. Let $x_i = i\Delta x$, $0 \leq i \leq N$ and \bar{x}_i the midpoint of $[x_i, x_{i+1}]$. For f defined on Λ_{2N} , define the forward difference operators

$$\begin{aligned} \Delta_i f &= f_{x_{i+1}} - f_{x_i} \\ \bar{\Delta}_i f &= f_{\bar{x}_i} - f_{x_i}, \quad 0 \leq i \leq N-1. \end{aligned}$$

For $x = x_k$, we have

$$f_x = \sum_{i=0}^{k-1} \Delta_i f.$$

and for any two functions f and g , we have

$$\Delta_i(fg) = (\Delta_i f)g_{x_i} + f_{x_i}\Delta_i g + \Delta_i f \Delta_i g \quad (3.30)$$

$$= (\Delta_i f)g_{\bar{x}_i} + f_{\bar{x}_i}\Delta_i g + \left(\Delta_i f \Delta_i g - (\bar{\Delta}_i f)\Delta_i g - \Delta_i f \bar{\Delta}_i g \right) \quad (3.31)$$

Equation (3.30) is an Ito-type formula for the differential of a product. Equation (3.31), which uses the midpoint rule, converts the Ito differentials in (3.30) into Stratonovich differentials.

We have

$$\begin{aligned} \bar{\Delta}_i g_{2N}^\alpha &= -M^\alpha([x_i, \bar{x}_i])g_{2N,x_i}^\alpha \\ \Delta_i g_{2N}^\alpha &= g_{2N,x_{i+1}}^\alpha - g_{2N,\bar{x}_i}^\alpha + g_{2N,\bar{x}_i}^\alpha - g_{2N,x_i}^\alpha \\ &= \left(-M^\alpha([\bar{x}_i, x_{i+1}])(-M^\alpha([x_i, \bar{x}_i]) + 1) - M^\alpha([x_i, \bar{x}_i]) \right) g_{2N,x_i}^\alpha \\ &= \left(-M^\alpha([x_i, x_{i+1}]) + M^\alpha([\bar{x}_i, x_{i+1}])M^\alpha([x_i, \bar{x}_i]) \right) g_{2N,x_i}^\alpha \end{aligned}$$

and similarly with $\Delta_i h_{2N}$ and $\bar{\Delta}_i h_{2N}$. Thus,

$$\begin{aligned} \bar{\Delta}_i h_{2N}^{-1} &= h_{2N,x_i}^{-1} \left(\left(1 - M^0([x_i, \bar{x}_i]) \right)^{-1} - 1 \right) \\ \Delta_i h_{2N}^{-1} &= h_{2N,x_i}^{-1} \left(\left(1 - M^0([x_i, x_{i+1}]) + M^0([\bar{x}_i, x_{i+1}])M^0([x_i, \bar{x}_i]) \right)^{-1} - 1 \right). \end{aligned}$$

For notational clarity, we temporarily drop the superscript α on the g_{2N}^α and \hat{g}_N^α below.

Making use of (3.31) we have

$$\Delta_i \hat{g}_N = \left((\Delta_i h_{2N}^{-1}) g_{2N, \bar{x}_i} + h_{2N, \bar{x}_i}^{-1} (\Delta_i g_{2N}) \right) h_{2N, x_-^\alpha} + R_i \quad (3.32)$$

where the remainder R_i is given by

$$R_i = h_{2N, x_i}^{-1} \left(-M^0([x_i, x_{i+1}]) M^\alpha([x_i, x_{i+1}]) + M^0([x_i, \bar{x}_i]) M^\alpha([x_i, x_{i+1}]) + \right. \quad (3.33)$$

$$\left. + M^0([x_i, x_{i+1}]) M^\alpha([x_i, \bar{x}_i]) + O([x_i, x_{i+1}]^3) \right) g_{2N, x_i} h_{2N, x_-^\alpha}. \quad (3.34)$$

Since the expectation of the terms quadratic in the M 's in R_i is zero, these terms are $O(N^{-2})$. It follows that $R_i = O(N^{-2})$. In what follows the precise nature of the remainder R_i may change from line to line; it is only required to be a function that is $O(N^{-2})$ and is of the form $O([x_i, x_{i+1}]^2)$ or $O([x_i, x_{i+1}]) \underline{Q}(N^{-1})$.

Consider the leading term of (3.32). We want to write $\Delta_i h_{2N}^{-1}$ and $\Delta_i g_{2N}$ in terms of the midpoint of the corresponding function in order to get differentials that are of Stratonovich type. Thus,

$$\begin{aligned} \Delta_i g_{2N} &= g_{2N, x_{i+1}} - g_{2N, \bar{x}_i} + g_{2N, \bar{x}_i} - g_{2N, x_i} \\ &= -M^\alpha([\bar{x}_i, x_{i+1}]) g_{2N, \bar{x}_i} + \left(1 - \left(1 - M^\alpha([x_i, \bar{x}_i]) \right)^{-1} \right) g_{2N, \bar{x}_i} \\ &= -M^\alpha([x_i, x_{i+1}]) g_{2N, \bar{x}_i} + R_i \\ \Delta_i h_{2N}^{-1} &= h_{2N, x_{i+1}}^{-1} - h_{2N, \bar{x}_i}^{-1} + h_{2N, \bar{x}_i}^{-1} - h_{2N, x_i}^{-1} \\ &= h_{2N, \bar{x}_i}^{-1} \left(\left(1 - M^0([\bar{x}_i, x_{i+1}]) \right)^{-1} - 1 \right) + h_{2N, \bar{x}_i}^{-1} \left(1 - \left(1 - M^0([x_i, \bar{x}_i]) \right) \right) \\ &= h_{2N, \bar{x}_i}^{-1} M^0([x_i, x_{i+1}]) + R_i. \end{aligned}$$

Here, we used the fact that terms quadratic in M^α or M^0 are $O(N^{-2})$, due to the isotropic condition $C^{\alpha\alpha} = C^{00} = 0$, so that we may shuffle them into the remainder R_i .

Thus (3.32) becomes

$$\begin{aligned} \Delta_i \hat{g}_N &= h_{2N, \bar{x}_i}^{-1} \left(M^0([x_i, x_{i+1}]) - M^\alpha([x_i, x_{i+1}]) \right) g_{2N, \bar{x}_i} h_{2N, x_-^\alpha} + R_i \\ &= -h_{2N, \bar{x}_i}^{-1} \tilde{M}^\alpha([x_i, x_{i+1}]) h_{2N, \bar{x}_i} (h_{2N, \bar{x}_i}^{-1} g_{2N, \bar{x}_i} h_{2N, x_-^\alpha}) + R_i \\ &= - \left((\text{ad}(h_{2N}) \circ \tilde{M}^\alpha)([x_i, x_{i+1}]) \right) \hat{g}_{N, \bar{x}_i} + R_i \\ &= - \left((\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)([x_i, x_{i+1}]) \right) \hat{g}_{N, \bar{x}_i} + R_i \end{aligned} \quad (3.35)$$

where we used Lemma 3.10 in the last line.

Equation (3.35) says \hat{g}_N satisfies a discretized Stratonovich differential equation up to a lower order remainder R_i . Since terms of order $O(N^{-2})$ are subdominant when looking at order Δx (the order of a single increment), we expect $\hat{g}_{N, x}$ to be close to

$$g_{N, x}^\sharp := 1 + \sum_{n=1}^{\infty} (-1)^n \sum_{\substack{i_+ \succ i_n \succ \dots \succ i_1 \\ I_{i_k} \in \tilde{\mathcal{I}}_N[x_-^\alpha, x]}} (\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)(J_{i_n}) \cdots (\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)(J_{i_1}). \quad (3.36)$$

which is a discretized path ordered exponential, whose individual terms are iterated Stratonovich sums. Since $g_{N,x}^\sharp$ is given by iterated Stratonovich sums, its increment satisfies

$$\Delta_i g_N^\sharp = -(\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)([x_i, x_{i+1}]) g_{N,\bar{x}_i}^\sharp,$$

a midpoint rule rather than a left endpoint rule.

We have

$$\begin{aligned} \hat{g}_{N,x_{i+1}} - g_{N,x_{i+1}}^\sharp &= (\hat{g}_{N,x_{i+1}} - \hat{g}_{N,x_i}) + (\hat{g}_{N,x_i} - g_{N,x_i}^\sharp) - (g_{N,x_{i+1}}^\sharp - g_{N,x_i}^\sharp) \\ &= \left(-(\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)([x_i, x_{i+1}]) \hat{g}_{N,\bar{x}_i} + R_i \right) \\ &\quad + (\hat{g}_{N,x_i} - g_{N,x_i}^\sharp) + \left((\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)([x_i, x_{i+1}]) \right) g_{N,\bar{x}_i}^\sharp \\ &= \left(-(\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)([x_i, x_{i+1}]) \right) (\hat{g}_{N,\bar{x}_i} - g_{N,\bar{x}_i}^\sharp) \\ &\quad + (\hat{g}_{N,x_i} - g_{N,x_i}^\sharp) + R_i. \end{aligned} \quad (3.37)$$

Instead of incrementing from x_i to x_{i+1} by step size Δx , we could have also incremented from \bar{x}_i to \bar{x}_{i+1} . In doing so, the above analysis can be repeated to show that

$$\begin{aligned} \hat{g}_{N,\bar{x}_{i+1}} - g_{N,\bar{x}_{i+1}}^\sharp &= \left(-(\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)([\bar{x}_i, \bar{x}_{i+1}]) \right) (\hat{g}_{N,x_{i+1}} - g_{N,x_{i+1}}^\sharp) \\ &\quad + (\hat{g}_{N,\bar{x}_i} - g_{N,\bar{x}_i}^\sharp) + R_i. \end{aligned} \quad (3.38)$$

Let $y_j = j\Delta x/2$ be an enumeration of the points x_i, \bar{x}_i of Λ_{2N} , $j = 0, \dots, 2N$. Letting

$$a_j = \hat{g}_{N,y_j} - g_{N,y_j}^\sharp, \quad (3.39)$$

$$c_j = -(\text{ad}(h_{4N}) \hat{\circ} \tilde{M}^\alpha)([y_{j-1}, y_{j+1}]) \quad (3.40)$$

then (3.37) and (3.38) imply that we have the recurrence relation

$$a_{j+2} = c_{j+1} a_{j+1} + a_j + r_{j+1} \quad (3.41)$$

where r_j denotes a remainder that is $O(N^{-2})$ and of the form $O([y_{j-1}, y_{j+1}]^2)$ or $O(N^{-1})O([y_{j-1}, y_{j+1}])$. Solving this recurrence relation with

$$a_0 = 0$$

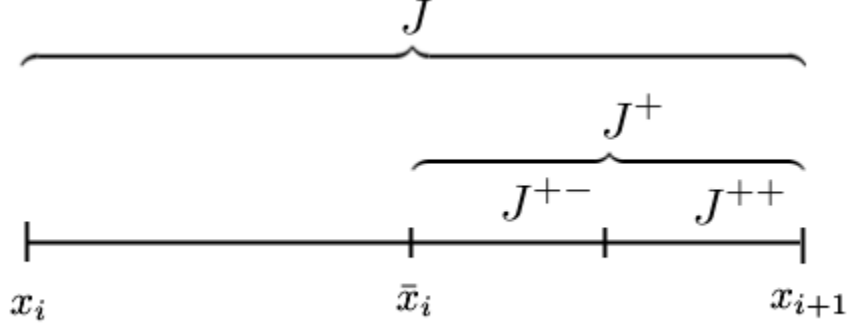
$$a_1 = r_0$$

we find that

$$a_j = \sum_{\bar{j}=0}^{j-1} P_{\bar{j}}^{(j)} r_{\bar{j}} \quad (3.42)$$

where $P_{\bar{j}}^{(j)}$ is a polynomial in c_1, \dots, c_j such that

- $\deg P_{\bar{j}}^{(j)} = j - 1 - \bar{j}$;
- individual monomial terms consist of products of distinct c_i 's, whose indices occur in descending order from left to right.

FIGURE 2. Nesting of intervals when $J = [x_i, x_{i+1}]$.

We want to show that $a_j = \hat{g}_{N,y_j} - g_{N,y_j}^\sharp$ is $O(N^{-1})$ for all $0 \leq j \leq 2k \leq 2N$. It suffices (and it is imperative) that we work at some fixed order ℓ in powers of λ as we let $N \rightarrow \infty$, since the presence of h_{4N}^{-1} terms occurring in $\text{ad}(h_{4N})$ forbid us to work at all orders uniformly in N . Thus, we need only consider terms of (3.42) that are at most of polynomial degree ℓ in the c_j 's. In what follows, estimates are uniform with respect to fixed ℓ .

For $d \leq \ell$, the number of (monic) monomials of degree d in the c_j is at most $O(N^d)$, since j ranges from $1, \dots, 2N - 1$. A generic term of the form

$$c_{j_d} \cdots c_{j_1} r_{j_0}, \quad j_d > \cdots > j_0 \quad (3.43)$$

is of order $O(N^{-(d+2)})$. This occurs when $j_{k+1} - j_k > 1$ for all $k = 0, \dots, d - 1$ so that the support of the infinitesimals occurring in the c_{j_k} and r_{j_0} are disjoint. For each adjacency $j_{k+1} = j_k + 1$, the order of (3.43) increases by a factor of N , but then the number of monomials with this condition also decreases by a factor of $O(N)$. So for each fixed \bar{j} in (3.42), considering all the monomials of $P_{\bar{j}}^{(j)}$ of degree d at most ℓ , we obtain an overall term that is $\ell \cdot O(N^{-2})$, since it is (essentially) a sum of $\ell \cdot O(N^d)$ terms of order $O(N^{-(d+2)})$. Summing over \bar{j} , we obtain $O(N)$ terms of order $\ell \cdot O(N^{-2})$, so that a_j is $O(N^{-1})$ for any fixed ℓ . Since ℓ is arbitrary, this shows that $\hat{g}_N = g_N^\sharp + O(N^{-1})$. \square

Let $\tilde{\mathcal{I}}_N^+ = \{J^+ : J \in \tilde{\mathcal{I}}_N\}$. From (3.17), we have

$$\tilde{\mathcal{I}}_N^+ = \mathcal{I}_{2N}. \quad (3.44)$$

For $J \in \tilde{\mathcal{I}}_N^+$, define

$$J^{++} := (J^+)^+ \quad (3.45)$$

$$J^{+-} := (J^+)^-. \quad (3.46)$$

These intervals are of size $\Delta x/4$ and partition $[0, L]$. Call the resulting set of intervals $\tilde{\mathcal{I}}_N^{++}$ and $\tilde{\mathcal{I}}_N^{+-}$, respectively. So we have

$$\mathcal{I}_{4N} = \tilde{\mathcal{I}}_N^{+-} \cup \tilde{\mathcal{I}}_N^{++}. \quad (3.47)$$

Observe that h_{4N} is an iterated Ito sum formed out of intervals belonging to \mathcal{I}_{4N} . For the next lemma, we make crucial use that h is defined on Λ_{4N} using Ito sums. (By comparison, in Lemma 3.11, h , like the g^α , could have been defined using the Stratonovich integral as in the second case of (3.23).

Lemma 3.12. (*Gauge-invariance*) *Let P be any polynomial in the $M^\alpha(J)$, with $J \in \tilde{\mathcal{I}}_N$, $0 \leq \alpha \leq m$. Regarding P as the induced polynomial in the $M^\alpha(J^+)$ as well, $J^+ \in \tilde{\mathcal{I}}_N^+$, let P^h be the corresponding polynomial with each $M^\alpha(J^+)$ replaced with $(\text{ad}(h_{4N}) \circ M^\alpha)(J^+)$. Then*

$$\mathbb{E}(\text{tr } P) = \mathbb{E}(\text{tr } P^h) \quad (3.48)$$

PROOF. Let V_N be the (real) vector space spanned by the variables

$$\mathcal{V}_N = \{M^{\alpha,a}(I) : 1 \leq \alpha \leq m, I \in \mathcal{I}_{2N}\} \cup \{M^{0,a}(\tilde{I}) : \tilde{I} \in \mathcal{I}_{4N}\}. \quad (3.49)$$

The expectation operator restricted to polynomial functions on V_N (i.e. polynomials in the variables appearing in \mathcal{V}_N) is completely determined by the covariance matrix

$$\mathcal{C} = \mathcal{C}_{(\alpha,I,a),(\beta,J,b)} := \mathbb{E}(M^{\alpha,a}(I)M^{\beta,b}(J)), \quad M^{\alpha,a}(I), M^{\beta,b}(J) \in \mathcal{V}_N.$$

Observe that \mathcal{C} is a tensor product of two matrices: one parametrizing the $0 \leq \alpha \leq m$ index and interval variables I and other the identity matrix with respect to Lie-algebra indices. In what follows, all our matrices factorize in this way into a non Lie-algebra part tensored the identity matrix on the Lie algebra.

The matrix \mathcal{C} is not necessarily invertible, but by adding an arbitrarily small matrix, call it ϵ , we can make $\mathcal{C} + \epsilon$ is invertible (since \mathcal{C} is weighted by λ , we do the same for ϵ). Let A_ϵ be its inverse. Define \mathbb{W}_{A_ϵ} to be the operator on power series functions on V_N given by applying the Wick rule (i.e. the rule (3.2)) using the covariance $\mathcal{C} + \epsilon$. If $A + \epsilon$ were positive-definite, then \mathbb{W}_{A_ϵ} could be expressed as integration against a Gaussian measure

$$d\mu_{A_\epsilon} = c_{A_\epsilon} e^{-(v, A_\epsilon v)/2} \prod_{X \in \mathcal{V}_N} dX,$$

where c_{A_ϵ} is a normalization constant and $(v, A_\epsilon v)$ stands for the pairing on V_N dual to the pairing $\mathcal{C} + \epsilon$ on V_N^* (which in the appropriate basis is given by the inverse matrix A_ϵ to $\mathcal{C} + \epsilon$). In other words,

$$\mathbb{W}_{A_\epsilon}(f) = \int_{\mathcal{V}_N} f(v) d\mu_{A_\epsilon}(v). \quad (3.50)$$

Thus, for \mathcal{C} positive-definite (and hence $\mathcal{C} + \epsilon$ for ϵ small) the expression on the right-hand side of (3.50), being an integral, is invariant under changes of coordinates via the usual properties of integrals. However, in [14], it is shown that the series in λ one obtains on the left-hand side of (3.50) using the Wick rule, i.e., the *Wick expansion*, is invariant under changes of coordinates. In other words, while we no longer have a Gaussian measure in general for the right-hand side of (3.50), formal calculus manipulations still hold in the sense that if $\Theta : V_N \rightarrow V_N$ is a diffeomorphism (or more generally, an invertible power series) that fixes the origin, then writing

$$\Theta^*(f(v) d\mu_{A_\epsilon}(v)) = \tilde{f}(v) d\mu_{\tilde{A}}(v)$$

where the right-hand side is expressed using the usual change of variables formula, we have

$$\mathbb{W}_{A_\epsilon}(f) = \mathbb{W}_{\tilde{A}}(\tilde{f}). \quad (3.51)$$

In short, (3.51) is a well-defined algebraic identity between two formal series in λ , without regard to there being an honest measure (though the expressions for \tilde{A} and \tilde{f} are obtained as though one were doing a change of variables for a measure.) See [14, Theorem 1.5] for a proof of (3.51). See also [15] for additional details.

We prove (3.48) from (3.51) by showing that the right-hand side is a change of coordinates in a Wick expansion, and hence, the result is unaffected. So consider the power series change of variables

$$\begin{aligned} \Theta : V_N &\rightarrow \widehat{\text{Sym}}(V_N) \\ X &\mapsto \text{ad}(h_{4N, x^*})X \end{aligned}$$

where if $X = M^\alpha(J)$, then

$$x^* = \begin{cases} \text{midpoint} & J \in \tilde{\mathcal{I}}_N^+ = \mathcal{I}_{2N} \\ \text{right endpoint} & J \in \tilde{\mathcal{I}}_N^{+-} \\ \text{left endpoint} & J \in \tilde{\mathcal{I}}_N^{++}. \end{cases}$$

This choice of x^* is so that if I and J are intervals from $\tilde{\mathcal{I}}_N^+ \cup \tilde{\mathcal{I}}_N^{+-} \cup \tilde{\mathcal{I}}_N^{++}$ that have overlapping interior, the corresponding x^* are equal. This is crucial in what follows and is ultimately the reason why our discretizations and interval subdivisions are as they are.

So we have

$$\int_{V_N} \text{tr}(P) d\mu_{A_\epsilon} = \int_{V_N} \Theta^* \left(\text{tr}(P) d\mu_{A_\epsilon} \right). \quad (3.52)$$

in the sense of Wick expansions. Now, $\Theta^* \text{tr}(P) = \text{tr}(P^h)$. Here, we used that

$$(\text{ad } h_{4N} \circ M^\alpha)(J^+) = \Theta(M^\alpha(J^{+-})) + \Theta(M^\alpha(J^{++})), \quad J \in \tilde{\mathcal{I}}_N, \quad 0 \leq \alpha \leq m.$$

It remains to show that Θ^* preserves $d\mu_{A_\epsilon}$. For then if we do that, letting $\epsilon \rightarrow 0$ in (3.52) yields (3.48).

Now $(\Theta(v), A_\epsilon \Theta(v)) = (v, A_\epsilon v)$ since the adjoint action preserves the inner product on the Lie algebra and our definition of x^* ensures that elements of \mathcal{V}_N which pair nontrivially always become conjugated by h_{4N} evaluated at matching points. (In Figure 2, x^* is the midpoint of \bar{x}_i and x_{i+1} for J^+ , J^{+-} , and J^{++} .) It remains to show that Θ preserves the Lebesgue measure $\prod_{X \in \mathcal{V}_N} dX$, i.e., the Jacobian matrix $\mathcal{J}(\Theta)$ has determinant equal to one. We will show that $\mathcal{J}(\Theta)_{I,J}$ is block lower triangular (the blocks being given by the suppressed curve indices and Lie-algebraic indices), with the diagonal blocks being unimodular. Here, we order our interval indices in ascending order as follows

$$J_1^-, J_1^+, J_1, J_2^-, J_2^+, J_2, \dots, J_{2N}^-, J_{2N}^+, J_{2N}$$

where $J_i = [(i-1)/2N, i/2N] \in \mathcal{I}_{2N}$.

For Θ acting on $X = M^\alpha(J)$, $1 \leq \alpha \leq m$, then $\Theta(X)$ is linear in X . Hence the derivative of $\Theta(X)$ with respect to X is just the adjoint action of h_{4N} , and this is unimodular on the J -block $\mathcal{J}(\theta)_{I,J}$ (the derivative of $\Theta(X)$ with respect to the M^0 -variables comprising h_{4N} will be in the strictly lower triangular part of $\mathcal{J}(\Theta)$). For the remaining case $X = M^0(J)$, $J \in \mathcal{I}_{4N}$, then $\Theta(X)$ is a formal power series in the $M^0(J')$, $J' \in \mathcal{I}_{4N}$, with $J' \leq J$ (in the

natural ordering of intervals). Remarkably it is linear in $M^0(J)$, and so the corresponding diagonal block $\mathcal{J}(\Theta)_{J,J}$ is unimodular. This is evident for $J \in \tilde{\mathcal{I}}_N^{++}$, since then conjugation by $\text{ad}(h_{4N,x^*})$ happens at the left-endpoint and so is comprised out of $M^0(J')$ with J' strictly preceding J . For $J \in \tilde{\mathcal{I}}_N^{+-}$, let $J = [x, x + \Delta/4]$. So $x^* = x + \Delta x/4$, and we have

$$h_{4N,x^*} = (1 - M^0(J))h_{4N,x}.$$

Hence,

$$\begin{aligned} \Theta(M^0(J)) &= \text{ad}(h_{4N,x^*})M^0(J) \\ &= \text{ad}(h_{4N,x})M^0(J) \end{aligned}$$

which is linear in $M^0(J)$. The lemma now follows. \square

4. PROOFS OF MAIN THEOREMS

We now prove our main theorems computing various kinds of expectations of the Wilson loop operator $W_{f,\gamma}(A)$. As before, γ is an arbitrary piecewise C^1 -closed curve which we may assume to be admissible, \mathfrak{g} is embedded inside $\text{End}(V)$, the space of matrices acting on some vector space V , and $f = \text{tr}$ (trace on $\text{End}(V)$). Without loss of generality, we assume the image of γ is contained within the strip $[0, L] \times \mathbb{R} \subset \mathbb{R}^2$. Write

$$\gamma \equiv \gamma_m \cdots \gamma_1$$

in terms of a horizontal curve decomposition. Let x_α^\pm be the initial and final x -coordinates of the horizontal γ_α , $1 \leq \alpha \leq m$. So these points satisfy the matching condition (3.22). Let $\gamma_0 : \mathbb{R} \rightarrow \mathbb{R}$ be the horizontal curve given by being the identity map along the x -axis in \mathbb{R}^2 .

Let

$$C^{\alpha\beta} = \sigma_{\alpha\beta} \bar{G}_{pax}(\bar{\gamma}_\alpha(x), \bar{\gamma}_\beta(x)) \quad (4.1)$$

where $\sigma_{\alpha\beta} = \pm 1$ according to whether γ^α and γ^β move in the same or opposite directions. Then

$$\tilde{C}^{\alpha\beta}(x) = C^{\alpha\beta}(x) - C^{\alpha 0}(x) - C^{0\beta}(x) + C^{00}(x) \quad (4.2)$$

$$= \sigma_{\alpha\beta} \bar{G}_{ax}(\bar{\gamma}^\alpha(x), \bar{\gamma}^\beta(x)). \quad (4.3)$$

Thus, the “random variables” $M^\alpha(I)$ and $\tilde{M}^\alpha(I)$ as defined in Definition 3.1 and equation (3.26) from the previous section capture how the integrals of A along $\gamma^\alpha|_I$ are “distributed” in partial axial-gauge and complete axial-gauge, respectively, with only complete axial-gauge truly giving rise to honest measure theoretic notions.

Let

$$P_N^\alpha = P_{N,x_-^\alpha \rightarrow x_+^\alpha}^{M^\alpha} \quad (4.4)$$

$$\tilde{P}_N^\alpha = \tilde{P}_{N,x_-^\alpha \rightarrow x_+^\alpha}^{\tilde{M}^\alpha} \quad (4.5)$$

using Definition 3.6 and 3.7. They represent discretized Ito and Stratonovich parallel transport along γ^α in partial and complete axial-gauge, respectively. (Note that since $C^{\alpha\alpha} = 0$, we could have used Stratonovich parallel transport in (4.4)).

We now prove Theorems 1 and 2.

Theorem 4.1. *Define*

$$\langle W_{f,\gamma} \rangle_{ax} = \lim_{N \rightarrow \infty} \mathbb{E}(\text{tr}(\tilde{P}_N^m \cdots \tilde{P}_N^1)) \quad (4.6)$$

Then

$$\langle W_{f,\gamma} \rangle_{ax} = \langle W_{f,\gamma} \rangle.$$

PROOF. In [8], it is shown how $\langle W_{f,\gamma} \rangle$, given by integrals involving heat kernels on G , can be computed by understanding the joint distribution of stochastic parallel transports of the constituent horizontal curves γ_α of γ . Namely, as a first step, if γ^α is a right-moving horizontal curve, then $\tilde{g}_t^\alpha := \tilde{P}_{N,x_-^\alpha \rightarrow t}^{\tilde{M}^\alpha}$ satisfies the Stratonovich differential equation

$$\begin{aligned} d\tilde{g}_t^\alpha + dW(\mathbf{1}_{R^\alpha(t)}) \circ \tilde{g}_t^\alpha &= 0 \\ \tilde{g}_{t_0}^\alpha &= 1 \end{aligned} \quad (4.7)$$

where $t_0 = x_-^\alpha$; W is two-dimensional \mathfrak{g} -valued white-noise; and $R^\alpha(t)$ is the region between the graph of $\tilde{\gamma}^\alpha|_{[x_-^\alpha, t]}$ and the x -axis.⁴ Thus, one of the main results of [8] is that

$$\langle W_{f,\gamma} \rangle = \mathbb{E}(\text{tr}(\tilde{g}^m \cdots \tilde{g}^1)) \quad (4.8)$$

where \mathbb{E} is an honest stochastic expectation.

From the work of [4], the solution to (4.7) is given by the usual path ordered exponential expansion, where all integrals are understood in terms of iterated Stratonovich integrals of the $dW(\mathbf{1}_{R^\alpha(t)})$. But from the definitions, we have $dW(\mathbf{1}_{R^\alpha(t)}) = \tilde{M}_t^\alpha$. So then the solution to (4.7) is given by

$$\tilde{g}_t^\alpha = 1 + \sum_{n=1}^{\infty} (-1)^n \int_{t \geq t_n \geq \cdots \geq t_1 \geq t_0} dM_{t_n} \circ \cdots \circ dM_{t_1}, \quad 1 \leq \alpha \leq m.$$

Next, Lemmas 3.5 and 3.8 show that

$$\mathbb{E}(\text{tr}(\tilde{g}^m \cdots \tilde{g}^1)) = \lim_{N \rightarrow \infty} \mathbb{E}(\text{tr}(\tilde{P}_N^m \cdots \tilde{P}_N^1)) \quad (4.9)$$

since the P_N^α are discretizations of the \tilde{g}_N^α . Thus, $\langle W_{f,\gamma} \rangle = \langle W_{f,\gamma} \rangle_{ax}$. \square

Note that the right-hand side of (4.9) is a priori a nonexplicit stochastic expectation. On the other hand, the right-hand side of (4.9), for each N , can be evaluated using the Wick rule. Letting $N \rightarrow \infty$, the expectation converges to a (complicated) Riemann integral yielding $\langle W_{f,\gamma} \rangle_{ax}$. See Remark 4.3 below for an example computation.

Proof of Theorem 2: We have

$$\langle W_{f,\gamma} \rangle_{pax} = \lim_{N \rightarrow \infty} \mathbb{E}(\text{tr}(P_N^m \cdots P_N^1)), \quad (4.10)$$

which one can either take as a definition or else see that it is equivalent to the definition we gave in the introduction involving Feynman diagrams using the partial axial-gauge propagator. Next, we can replace each P_N^α , which is the element $g_{N,x_+^\alpha}^\alpha$ given by (3.23), with $\hat{g}_{N,x_+^\alpha}^\alpha$ as defined in (3.25). We then replace $\hat{g}_{N,x_+^\alpha}^\alpha$ with the leading term of (3.29) by Lemma 3.11,

⁴This assumes the graph of $\tilde{\gamma}^\alpha$ lies above the x -axis. However, since partial axial-gauge and the lattice formulation are translation-invariant, we can suppose γ and hence the γ_α all lie above the x -axis.

since the terms of order $O(N^{-1})$, as defined by Definition 3.4, vanish in the limit $N \rightarrow \infty$. Next, we apply Lemma 3.12 to eliminate all the $\text{ad}(h_{4N})$ terms in the leading terms of (3.29) when we compute the expectation. But once we remove these terms, what we have left is Stratonovich parallel transport with respect to the \tilde{M}^α , i.e., we obtain

$$\langle W_{f,\gamma} \rangle_{pax} = \lim_{N \rightarrow \infty} \mathbb{E}(\text{tr}(\tilde{P}_N^m \cdots \tilde{P}_N^1)).$$

This shows $\langle W_{f,\gamma} \rangle_{pax} = \langle W_{f,\gamma} \rangle_{ax}$. \square

Remark 4.2. Suppose we insert a mass m^2 into the Yang-Mills action in partial axial-gauge

$$\frac{1}{2\lambda} \int dx dy \left(\langle \partial_y A_x, \partial_y A_x \rangle + m^2 \langle A_x, A_x \rangle \right) \quad (4.11)$$

Replacing the action in (2.3) with the one above, we obtain a bona fide Gaussian measure whose covariance is λ times

$$G_m(x - x', y - y') = \frac{e^{-m|y-y'|}}{2m} \delta(x - x'). \quad (4.12)$$

Note that $\frac{1}{2m}e^{-m|y-y'|}$ is the unique Green's function for $-\partial_y^2 + m^2$ that is invariant under translations and reflections. Thus, we obtain a measure which is Ornstein-Uhlenbeck measure in the y -direction and white-noise in the x -direction. Unfortunately, the $m^2 \rightarrow 0$ limit of (4.12) does not exist. This is because the limit kinetic operator has constant zero modes, which if taken into account, leads us to consider our massive Green's function renormalized by an additive constant:

$$\lim_{m \rightarrow 0} \frac{e^{-m|y-y'|} - 1}{2m} = -\frac{1}{2}|y - y'|. \quad (4.13)$$

This latter expression yields the partial axial-gauge Green's operator.

We can try to analyze stochastic holonomy with respect to (4.12) and let $m \rightarrow 0$. Unfortunately, for nonabelian gauge group G , it is unclear whether the limiting (expectation of) holonomy exists. For G abelian, we were able to organize Feynman diagrams in a way that exhibits the zero mode subtraction (4.13). Unfortunately, when G is nonabelian, the noncommutativity of the basis elements e_a of \mathfrak{g} complicates the combinatorics involved in showing that singular elements cancel in the massless limit, see the next remark. Being unable to control such combinatorics for general curves to arbitrary order, we instead developed the algebraic stochastic methods in this paper. In some sense, our algebraic use of gauge-invariance in Lemma 3.12 circumvents such difficult combinatorics.

Remark 4.3. The subtraction (4.13) also suggests how Feynman diagrams should be organized when equating $\langle W_{f,\gamma} \rangle_{ax}$ with $\langle W_{f,\gamma} \rangle_{pax}$. Let us sketch how to compute $\langle W_{f,\gamma} \rangle_{ax}$ for the example in the introduction. We proceed somewhat formally using the direct expression (2.9) to compute integrals; these statements can be justified by using the rigorous stochastic definition (4.6).

We have two types of Wick contractions for $\langle W_{f,\gamma} \rangle_{ax}$. As before, we have those which are vertical “chords” joining a pair points on γ_1 and γ_3 . But now we also have “tadpoles” joining adjacent pairs of points on either γ_1 or γ_3 (which makes their x -coordinates collapse). (If we Wick contract nonadjacent points of γ_1 and γ_3 , the path ordering condition makes

intermediate points range over a set of measure zero, and thus we obtain zero.) Tadpoles contribute a factor of $+\frac{1}{2}\bar{G}_{ax}(y, y) = \frac{|y|}{2}$; the plus sign arises because we are joining two points along the same curve, and the $\frac{1}{2}$ arises from a careful analysis of the Stratonovich midpoint rule. Thus, ignoring Lie-algebraic factors for the moment, we can organize Wick contractions into triplets consisting of a tadpole on a double point t_i belonging to γ_1 , a chord joining t_i to t_{i+n} , and a tadpole on a double point t_{i+n} belonging to γ_3 . (Thus, in Figure 1, we decorate each chord with a tadpole and obtain a “dumbbell”.) The scalar part of each such dumbbell yields

$$-\bar{G}_{ax}(\gamma_+(x_i), \gamma_-(x_i)) + \frac{1}{2}\bar{G}_{ax}(\gamma_-(x_i), \gamma_-(x_i)) + \frac{1}{2}\bar{G}_{ax}(\gamma_+(x_i), \gamma_-(x_i)) = \frac{1}{2}(\gamma_+(x_i) - \gamma_-(x_i)),$$

exactly coinciding with the scalar contribution of the partial axial-gauge propagator. Fortunately, for the simple convex curve we have drawn, the Lie-algebraic factors all become powers of the quadratic Casimir, and so all scalar factors are weighted equally. Thus, the above computation implies that $\langle W_{f,\gamma} \rangle_{ax} = \langle W_{f,\gamma} \rangle_{ax}$ at every order in λ .

However, as soon as γ begins to wind nontrivially or self-intersect, we obtain nontrivial Lie-algebraic factors that weight various Wick contractions. It becomes nontrivial to organize these factors in a way that exhibits the equality between $\langle W_{f,\gamma} \rangle_{ax}$ and $\langle W_{f,\gamma} \rangle_{ax}$. If we had used the mass regulated propagator (4.12), such Lie-theoretic factors are the source of the combinatorial difficulty involved in showing that a massless limit exists.

REFERENCES

- [1] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão, and T. Thiemann. *SU(N) quantum Yang-Mills theory in two dimensions: A complete solution*. J. Math. Phys. 38 (1997), 5453–5482.
- [2] L. Arnold. *Stochastic differential equations: theory and applications*. Wiley-Interscience, New York-London-Sydney, 1974.
- [3] A. Bassetto and G. Nardelli. *(1 + 1)-dimensional Yang-Mills theories in light cone gauge*. Int. J. Mod. Phys. A12 (1997) 1075–1090, Int. J. Mod. Phys. A12 (1997) 2947.
- [4] Ben Arous, G. Flots et séries de Taylor stochastiques. Probab. Theory Related Fields 81 (1989), no. 1, 29–77.
- [5] F. Brown. *Iterated integrals in quantum field theory*. Geometric and topological methods for quantum field theory, 188240, Cambridge Univ. Press, Cambridge, 2013.
- [6] S. Cordes, G. Moore, and S. Ramgoolam. Lectures on 2D Yang-Mills theory, equivariant cohomology, and topological field theories. [arxiv:hep-th/9411210](https://arxiv.org/abs/hep-th/9411210).
- [7] G. da Pratto. *An introduction to infinite-dimensional analysis*. Springer, Berlin, 2006.
- [8] B. Driver. *YM₂: continuum expectations, lattice convergence, and lassos*. Comm. Math. Phys. 123 (1989), no.4, 575–616.
- [9] B. Driver, F. Gabriel, B. Hall, and T. Kemp. *The Makeenko-Migdal equation for Yang-Mills theory on compact surfaces*. [arxiv:1602.03905](https://arxiv.org/abs/1602.03905)
- [10] D. Fine. *Quantum Yang-Mills on a Riemann surface*. Comm. Math. Phys. 140 (1991), no. 2, 321338.
- [11] L. Gross, C. King, and A. Sengupta. *Two-dimensional Yang-Mills theory via stochastic differential equations*. Ann. Physics 194 (1989), no. 1, 65–112.
- [12] T. Levy. *Yang-Mills measure on compact surfaces*. Mem. A.M.S. 166 (2003), no. 790.
- [13] A. A. Migdal. *Recursion equations in gauge field theories*. Sov. Phys. JETP 42, 3 (1975), 413–418.
- [14] T. Nguyen. *The perturbative approach to path integrals: A succinct mathematical treatment*. J. Math. Phys. 57, 092301 (2016)
- [15] T. Nguyen. *The perturbative approach to path integrals*. Animated video available at <https://youtu.be/QTjmLBzAdAA>

- [16] T. Nguyen. *Quantum Yang-Mills theory in two dimensions: Exact versus perturbative*. [arxiv:1508.06305](#)
- [17] T. Nguyen. *Wilson loop area law for 2D Yang-Mills in generalized axial gauge*. [arxiv:1601.04726](#)
- [18] A. Sengupta. *The Yang-Mills measure for S^2* . J. Fun. Anal. 108 (1992), 231–273.
- [19] I. M. Singer. *On the master field in two dimensions*. Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), 263–281. Progr. Math., 131, Birkhuser Boston, Boston, MA, 1995.
- [20] E. Witten. *On quantum gauge theories in two dimensions*. Comm. Math. Phys. 141 (1991), 153–209.

E-mail address: timothy@math.msu.edu

MICHIGAN STATE UNIVERSITY